Celebrating 60 YEARS of Fuzzy Sets

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Celebrating 60 Years of Fuzzy Sets

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Editorial: Celebrating 60 Years of Fuzzy Sets

Welcome to this special volume, a unique collection of insights and reflections from the global fuzzy set community. As we gather for Fuzz-IEEE 2025, we're not just attending a conference; we're commemorating 60 years since the inception of fuzzy sets, a revolutionary concept that forever changed how we model and understand uncertainty.

This volume is a testament to the enduring impact of Lotfi Zadeh's foundational work. It serves as a dynamic forum, inviting voices from across the globe to share their perspectives on the journey of fuzzy sets. We asked participants to reflect on what they consider the most significant recent contributions in fuzzy systems over the past decade, to identify the most pressing challenges looming for the future, and to highlight the truly pivotal contributions from these past six decades that hold personal significance.

The response has been remarkable, offering a rich tapestry of viewpoints that span historical milestones, cutting-edge advancements, and visionary outlooks. The diversity of contributions underscores the pervasive influence of fuzzy sets. Equally compelling are the thoughtful discussions on future challenges.

Several contributions have been devoted to recall the history of fuzzy sets, its evolution, achievements and conflicts. In some cases, it is the authors' personal history in the field that has been the main focus of the work. These narratives, whether historical or personal, are crucial for understanding the profound and global influence that fuzzy sets have exerted on the advancement of computational intelligence and the resolution of complex problems across diverse disciplines.

Beyond the core principles, another significant set of contributions in this volume delves into the historical evolution, current trends, and inherent challenges within specific topics and application areas of fuzzy sets. Several of these stand out as prominent areas of current research and development. Key application domains include Fuzzy Control, Healthcare, Biomedical Engineering, and Computational Social Sciences, among others.

The authors also highlight crucial research topics and tools that are shaping the future of the field. These encompass the burgeoning areas of Generative AI and Large Language Models (LLMs), Semantic Web Technologies, Information Fusion, Human-Centric AI and Explainable AI (XAI). Methodologies such as Genetic and Evolutionary Fuzzy Systems and the application of fuzzy sets in established paradigms like Machine Learning and Data Mining are also examined. Fundamental theoretical constructs like Possibility Theory, Representations by Levels, and Aggregation Functions are revisited, showcasing the ongoing intellectual dynamism driving fuzzy set theory forward.

To conclude, this compilation is more than just a historical record; it's a living dialogue. It represents the collective wisdom and foresight of a community dedicated to pushing the boundaries of intelligence and decision-making under uncertainty. We believe this open-access booklet will not only be a valuable resource for researchers and practitioners but also an inspiration for the next generation of innovators.

We extend our sincere gratitude to all who contributed their valuable insights, making this volume a vibrant celebration of fuzzy sets' rich past, dynamic present, and promising future. Your collective voices provide a compelling roadmap for the exciting journey ahead.

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Jesús Chamorro and Daniel Sánchez, editors

Fuzzy systems over the years

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***Abstract*—The purpose of this paper is to celebrate the richness of research and developments on fuzzy set theory in honor of the 60th anniversary of its invention by Lotfi Zadeh.**

***Index Terms*—fuzzy sets, fuzzy logic, fuzzy systems, history, applications.**

I. Introduction

Fuzzy sets were introduced 60 years ago by Lotfi Zadeh and they have undergone spectacular growth, essentially because of a number of large-scale industrial applications. To take the example of France, D. Willaeys and N. Malvache [47] continued E.H. Mamdani's seminal work [34] on fuzzy control in 1977 and they proposed to use it in a case study. They were not followed by industry and they had rapidly to stop their work on the topic. It was only after the successful utilization of fuzzy control by Japanese companies at the end of the 80s that French companies discovered the potential power of fuzzy systems and decided to use them in all possible fields at the beginning of the 90s. In the meantime, various real-world applications of fuzzy set theory were developed all over the world, Medical applications [2], and decision-making were among the most popular domains, and this is why it could grow and reach the Japanese engineers and researchers.

II. Recent impact of fuzzy systems

The past ten years have seen the emergence and supremacy of deep learning, which has focused on all types of applications dealing with data science. Therefore, I think that we can consider that the main impact of fuzzy methods in these recent years was mainly related to the utilization of the capacity of fuzzy sets to provide interfaces between numerical and linguistic values of variables, providing interesting ways of being easily understood by users in the framework of big data analysis.

The development of XAI, after the DARPA incentive in 2016 [14], put the light on a wide field in which the long-standing research on the expressiveness of fuzzy set-based methods could be developed and used. The natural capacity of a user to understand a fuzzy model, to use it, to understand how the outcomes are obtained, and finally to trust it, has been pointed out well before the movement in favour of XAI [6], [11], [12], [28]. The balance between complexity, accuracy, understandability and semantic interpretability is at the core of all learning methods dealing with big data [10], [16]. Multi-objective optimization strategies to maximize the interpretability while looking for high accuracy have been presented [33]. Experimental approaches to interpretability of fuzzy systems have been proposed as early as 2009 [3].

Subjectivity is an important component of the interactions between a system and its users, and its management by means of fuzzy sets has been studied for years [8], [35]. It now includes the subjective perception of explainability [43], which opens the door to a fruitful utilization of a fuzzy set-based knowledge representation. Explainability, understandability, expressiveness, and interpretability are various facets of the acceptability by the user of decisions made by artificial intelligence-based systems and it is clear that fuzzy methods provide an easy way to express and to handle them.

Another field of big data analysis where fuzzy sets are efficient is the summarization of data. The method has been introduced many years ago [4], [9], [23], [24], [48], but it found its full power with the capacity to cope with complexity and to provide simple scalable tools not requiring either parameters or hypothesis [25], [37], [44].

An important aspect of the current flow of news and information is the veracity of data, their reliability and the evaluation of uncertainty about the validity of information. Solutions have already been proposed by means of fuzzy and possibilistic rating methods and possibilistic logic [30], [31].

III. Fuzzy systems in the future

Given the overwhelming development of large language models and other automatic systems based on neural networks, deep learning, transformers and other methods to manage big data and to interact with users in a very natural way, it seems that the future of fuzzy systems lies on a collaboration with them. Hybrid systems have been used for a long time [38]. They have obviously taken various forms over the years and they adapt to the environment and the current streams. They can bring their capacity to manage subjective information and to consider gradual categories. Their ability to handle complexity makes them good candidates to deal with big data summarization, in particular in the case of temporal data. The management of subjective information is also promising. Emotional computing should incorporate more elements of fuzzy logic in the future. Robotics will certainly find new powerful utilisations of fuzzy logic.

IV. MAIN CONTRIBUTIONS OF FUZZY SET-BASED METHODS

Applications of fuzzy-set based methods have been so diverse that it is difficult to rank them. The major impact of fuzzy methods has been observed in industry and large-scale real-world applications. Fuzzy control has been important in the visibility of the power of fuzzy methods, but other aspects such as image processing have also been noteworthy.

The utilization of fuzzy systems by the NASA was for instance a major advancement. Patents were registered in image processing to track unfamiliar objects in videos and films or to detect debris during launch [45]. Fuzzy methods were used in autonomous orbital operations to cope with imprecise measurements from sensors [29] or for the docking of two geostationary satellites [39]. Many other utilizations have been described by NASA.

The most visible industrial application was probably the automatic subway train operation system based on predictive fuzzy control in the city of Sendai (Japan) [40]. This successful large-scale utilization of fuzzy control decided other Japanese companies to use fuzzy systems to solve their problems. We can cite for instance the use of fuzzy control for the design and implementation of a group of elevators [27] or the control of the automatic transmission shift schedule in cars by Nissan [49]. A wide field emerged at the same time with the use of fuzzy methods in robotics, for instance robots playing ping-pong game [18], interacting physically with humans [22] or through emotion understanding [13], for instance.

A little later, we cannot forget the development of an unmanned helicopter by M. Sugeno and his colleagues [46], with a semi-autonomous flight achieved on the basis of macroscopic flight commands given orally from the ground, which proved that fuzzy control can be used to construct safe and complex systems. Successful applications in Japan paved the way for other real-world applications around the world, for instance the fuzzy control of a nuclear system [21], [42]. To take the example of France, industrial applications started at the beginning of the 90s, in all directions, with regard to car makers [36], traffic routing in telephone networks [26], medical images [5], [41], air traffic flow management [50], [51], automatic subway traffic control [17] in particular.

It is worth noting that fuzzy logic was applied to all possible fieds [7], [19]. It is impossible to give an exhaustive list of these fields. Home appliances were the most popular applications, with air-conditioning systems, cameras, washing machines, ovens, blood pressure monitors, etc. Components of industrial plants were also handled with the help of fuzzy logic, for instance industrial furnaces, steel mills or semiconductor manufacturing. [15]. Fuzzy systems were also used in a military environment for the analysis of political crisis, air mission planning, the identification of on-site military activities for instance.

It was also used in the recognition or simulation of emotions in various environments: for robots, on social media or for business intelligence. Fuzzy logic was successfully used to process physiological signals for video game manufacturers [32], for brain-computer interfaces [1] for instance.

A wide range of applications concerns medical diagnosis, including many on the basis of image processing.

V. CONCLUSION

The acceptance of this new knowledge representation by users has been difficult when it emerged from the brain of a specialist of control theory, Lotfi Zadeh. During years, there was an opposition of experts in existing fields: probability theory, control theory, classical logic, linguistics. All of them felt that their own theory was sufficient to solve all problems in the world and they did not need a softened version of what they knew. It is only because applications started progressively to appear and proved to be successful that the acceptance of this different theory began to spread.

The importance of theoretical works which started very early everywhere in the world was of course a fundamental component of the success of fuzzy set theory. Without important rigorous basis, no applications would have been possible. But it is only thanks to real-world applications that the visibility of its potential increased. The first applications to medical diagnosis, economy or decision-making were interesting, but it is really through fuzzy control that a wide range of utilisations became possible. After innovative efforts from Japanese companies and academics working together, more companies were convinced of the power of these methods in Japan, which persuaded companies around the world to try these revolutionary techniques. It was very trendy to use ”fuzzy logic”, as was called the use of fuzzy set-based methods from this time. It was trendy to announce it publicly. When the fashion declined, the utilization of fuzzy logic continued, but it was less emphatically announced. It is now considered as a normal component of many systems and it does not need to be claimed, similarly to probability theory for instance.

In parallel, the explosion of new theoretical innovation faded, and the number of fundamental developments obviously decreased, as most of the necessary concepts have been created and analysed. So the situation is now stable, with continued research and development on fuzzy logic. It will be again in the light when a surprising application will attract public attention. We hope that the modern digital environment opens the way to such an application.

To conclude, we want to emphasize the fact that, since 1965, fuzzy set theory has offered a new way of thinking in mathematics. The introduction of this theory threw a spanner in the works by proposing gradual concepts, where they were previously, for long, considered as binary. By working on this, developing, exploiting, or creating new mathematical tools to handle fuzzy sets or their extensions, the community of fuzzy researchers has contributed to open the door to a new way of thinking. Now, in several domains, we can see the introduction of ”fuzzy-like” thought models, which also illustrates the contributions of this theory from a more conceptual point of view. For instance, uncertainty is now considered in machine learning of a more complex nature than only aleatoric, and it can also be epistemic [20].

References

[1] R. Abiyev, N. Akkaya, E. Aytac, I. Günsel, and A. Çağman. Brain-computer interface for control of wheelchair using fuzzy neural networks. *BioMed research international*, 2016.

[2] K.-P. Adlassnig. Fuzzy set theory in medical diagnosis. *IEEE Transactions on Systems, Man, and Cybernetics*, 16:260–265, 1986.

[3] J. M. Alonso, L. Magdalena, and G. González-Rodríguez. Looking for a good fuzzy system interpretability index: An experimental approach. *International Journal of Approximate Reasoning*, 51(1):115–134, 2009.

[4] F. E. Boran, D. Akay, and R. R. Yager. An overview of methods for linguistic summarization with fuzzy sets. *Expert Systems with Applications*, 61:356–377, 2016.

[5] S. Bothorel, B. Bouchon-Meunier, and S. Muller. A fuzzy logic based approach for semiological analysis of microcalcifications in mammographic images. *Int. J. Intell. Syst.*, 12(11-12):819–848, 1997.

[6] B. Bouchon-Meunier. Interpretability, a silver lining to a fuzzy cloud. In E. F. Magdalena L., Verdegay J., editor, *Enric Trillas: A Passion for Fuzzy Sets*, volume 322, pages 223–231. Springer, Cham, 2015.

[7] B. Bouchon-Meunier, M. Detyniecki, M.-J. Lesot, C. Marsala, and M. Rifqi. Real-world fuzzy logic applications in data mining and information retrieval. In P. P. Wang, D. Ruan, and E. E. Kerre, editors, *Fuzzy Logic: A Spectrum of Theoretical & Practical Issues*, pages 219–247. Springer Berlin Heidelberg, Berlin, Heidelberg, 2007.

[8] B. Bouchon-Meunier, M.-J. Lesot, and C. Marsala. Modeling and management of subjective information in a fuzzy setting. *International Journal of General Systems*, 42(1):3–19, 2013.

[9] B. Bouchon-Meunier and G. Moyse. Fuzzy linguistic summaries: Where are we, where can we go? In *IEEE Int. Conf. on Computational Intelligence for Financial Engineering & Economics (CIFEr)*, pages 317–324, 2012.

[10] B. Bouchon-Meunier and C. Marsala. Fuzzy modifiers at the core of interpretable fuzzy systems. In A. K. D. Tamir, N. Rishe, editor, *Fifty Years of Fuzzy Logic and its Applications*, volume 326, pages 51–63. Springer, Studies in Fuzziness and Soft Computing, 2015.

[11] B. Bouchon-Meunier. Interpretable decisions by means of similarities and modifiers. In *IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*, pages 524–529, 2009.

[12] J. Casillas, O. Cordón, F. Herrera, and L. Magdalena. Interpretability improvements to find the balance interpretability-accuracy in fuzzy modeling: An overview. In C. J., C. O., H. F., and M. L., editors, *Interpretability Issues in Fuzzy Modeling*, volume 128, pages 3–22. Springer, Heidelberg, 2003.

[13] L. Chen, M. Wu, M. Zhou, Z. Liu, J. She, and K. Hirota. Dynamic emotion understanding in human–robot interaction based on two-layer fuzzy svr-ts model. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 50(2):490–501, 2020.

[14] DARPA. https://www.darpa.mil/program/explainable-artificial-intelligence. 2016.

[15] Fuji-Electric. https://www.fujielectric.fr/en/blog/the-fundamentals-of-fuzzy-logic/.

[16] M. Gacto, R. Alcalá, and F. Herrera. Interpretability of linguistic fuzzy rule-based systems: an overview of interpretability measures. *Information Sciences*, 181:4340–4360, 2011.

[17] R. Hartani, S. Sellam, B. Bouchon-Meunier, and P. Gallinari. Modelling of an automatic subway traffic control by fuzzy logic and neural networks theory. *IFAC Proceedings Volumes*, 27(12):191–196, 1994.

[18] K. Hirota, Y. Arai, and S. Hachisu. Fuzzy controlled robot arm playing two-dimensional ping-pong game. *Fuzzy Sets and Systems*, 32(2):149–159, 1989.

[19] K. Hirota and M. Sugeno. *Industrial Applications of Fuzzy Technology in the World*. World Scientific, 1995.

[20] E. Hüllermeier and W. Waegeman. Aleatoric and epistemic uncertainty in machine learning: An introduction to concepts and methods. *Machine learning*, 110(3):457–506, 2021.

[21] C. Irvine. Power control of safe reactor using fuzzy logic. *AIP Conference Proceedings*, 608(1):711–716, 2002.

[22] M.-S. Ju, C.-C. Lin, D.-H. Lin, I.-S. Hwang, and S.-M. Chen. A rehabilitation robot with force-position hybrid fuzzy controller: hybrid fuzzy control of rehabilitation robot. *IEEE Transactions on Neural Systems and Rehabilitation Engineering*, 13(3):349–358, 2005.

[23] J. Kacprzyk and R. Yager. Linguistic summaries of data using fuzzy logic. *International Journal of General Systems*, 30(2):133–154, 2001.

[24] J. Kacprzyk and S. Zadrozny. Linguistic database summaries and their protoforms: towards natural language based knowledge discovery tools. *Information Sciences*, 173:281–304, 2005.

[25] K. Kaczmarek-Majer and O. Hryniewicz. Application of linguistic summarization methods in time series forecasting. *Information Sciences*, 478:580–594, 2019.

[26] J. Khalfet, P. Chemouil, and B. Bouchon-Meunier. Application of fuzzy control to adaptive traffic routing in telephone networks. In *Quardet'93*, Barcelona, Spain, 1993.

[27] C. Kim, K. Seong, H. Lee-Kwang, and J. Kim. Design and implementation of a fuzzy elevator group control system. *IEEE Transactions on Systems, Man, and Cybernetics - Part A: Systems and Humans*, 28(3):277–287, 1998.

[28] A. Laurent, C. Marsala, and B. Bouchon-Meunier. Improvement of the interpretability of fuzzy rule based systems: Quantifiers, similarities and aggregators. In A. R. J. Lawry, J. Shanahan, editor, *Modelling with Words*, pages 102–123. Springer-Verlag, Lecture Notes in Artificial Intelligence, 2003.

[29] R. N. Lea and Y. Jani. Fuzzy logic in autonomous orbital operations. *International Journal of Approximate Reasoning*, 6(2):151–184, 1992.

[30] M.-J. Lesot, O. Couchariere, B. Bouchon-Meunier, and J.-L. Rogier. Inconsistency degree computation for possibilistic description logic: an extension of the tableau algorithm. In *Annual Meeting of the North American Fuzzy Information Processing Society, NAFIPS'08*, 2008.

[31] M.-J. Lesot, T. Delavallade, F. Pichon, H. Akdag, B. Bouchon-Meunier, and P. Capet. Proposition of a semi-automatic possibilistic information scoring process. In *7th conf. of the European Society for Fuzzy Logic and Technology*, pages 949–956. Atlantis Press, 2011.

[32] F. Levillain, J. Orero, M. Rifqi, and B. Bouchon-Meunier. Characterizing player's experience from physiological signals using fuzzy decision trees. In *Proceedings of the 2010 IEEE Conference on Computational Intelligence and Games*, pages 75–82, Copenhagen, Denmark, 2010.

[33] L. Magdalena. Do hierarchical fuzzy systems really improve interpretability? In M. J. et al., editor, *Information Processing and Management of Uncertainty in Knowledge-Based Systems. Theory and Foundations. IPMU 2018*, volume 853, Cadix, Spain, 2018. Communications in Computer and Information Science, Springer.

[34] E. Mamdani and S. Assilian. An experiment in linguistic synthesis with a fuzzy logic controller. *International Journal of Man-Machine Studies*, 7:1–13, 1975.

[35] C. Marsala and B. Bouchon-Meunier. Fuzzy data mining and management of interpretable and subjective information. *Fuzzy Sets and Systems*, 281:252–259, 2015.

[36] X. Ménage. *Système flou pour le contrôle de qualité*. PhD thesis, Université P. & Marie Curie, Paris, 1996.

[37] M.-J. Lesot, G. Moyse, and B. Bouchon-Meunier. Interpretability of fuzzy linguistic summaries. *Fuzzy Sets and Systems*, 292:307–317, 2016.

[38] D. Nauck. Neuro-fuzzy systems: review and prospects. In *In Proceedings of Fifth European Congress on Intelligent Techniques and Soft Computing (EUFIT'97)*, pages 1044–1053. Citeseer, 1997.

[39] G. Ortega. Fuzzy logic techniques for rendezvous and docking of two geostationary satellites. *Telematics and Informatics*, 12(3):213–227, 1995. Advanced Space Technologies For Systems Autonomy.

[40] H. Oshima, S. Yasunobu, and S.-i. Sekino. Automatic train operation system based on predictive fuzzy control. In *Proceedings of the International Workshop on Artificial Intelligence for Industrial Applications*, pages 485–489, 1988.

[41] A. Rick, S. Bothorel, B. Bouchon-Meunier, S. Muller, and M. Rifqi. Fuzzy techniques in mammographic image processing. In E. E. Kerre and M. Nachtegael, editors, *Fuzzy Techniques in Image Processing*, pages 308–336. Physica-Verlag HD, Heidelberg, 2000.

[42] D. Ruan. Fuzzy logic in the nuclear research world. *Fuzzy Sets and Systems*, 74(1):5–13, 1995.

[43] A. Silva, M. Schrum, E. Hedlund-Botti, N. Gopalan, and M. G. and. Explainable artificial intelligence: Evaluating the objective and subjective impacts of xai on human-agent interaction. *International Journal of Human–Computer Interaction*, 39(7):1390–1404, 2023.

[44] G. Smits and M.-J. Lesot. Copils: Comparison of linguistic summaries. In *IPMU'2024*, pages 38–50. Springer Nature, 2024.

[45] SpaceNews. Nasa grants new imaging software patent and copyright license. *https://spacenews.com/nasa-grants-new-imaging-software-patent-and-copyright-license/*, June 22, 2004.

[46] M. Sugeno, I. Hirano, S. Nakamura, and S. Kotsu. Development of an intelligent unmanned helicopter. In *Proceedings of 1995 IEEE International Conference on Fuzzy Systems*, volume 5, pages 33–34, 1995.

[47] D. Willaeys, N. Malvache, and P. Hammad. Utilisation of fuzzy sets for system modelling and control. pages 1435–1439. IEEE, 1977.

[48] R. Yager. A new approach to the summarization of data. *Information Sciences*, 28:69–86, 1982.

[49] H. Yamaguchi, Y. Narita, H. Takahashi, and Y. Katou. Automatic transmission shift schedule control using fuzzy logic. *SAE Technical Paper 930674*, https://doi.org/10.4271/930674, 1993.

[50] L. Zerrouki, B. Bouchon-Meunier, and R. Fondacci. A fuzzy-neural model for co-ordination in air traffic flow management. In *International Workshop on Fuzzy Logic in Artificial Intelligence*, pages 190–204. Springer, 1997.

[51] L. Zerrouki, R. Fondacci, B. Bouchon-Meunier, and S. Sellam. Artificial intelligence techniques for coordination in air traffic flow management. In *FAC/IFIP/IFORS Symposium on Transportation Systems*, Chania, Greece, 1997.

Origins of Three Fuzzy Societies

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***Abstract*— This note describes my involvement with the birth of three fuzzy societies: the North American Fuzzy Information Processing Society (NAFIPS); the International Fuzzy Systems Association (IFSA); and the fuzzy component of the IEEE Computational Intelligence Society.**

I. How it all began

The idea of forming a professional society dedicated to the advancement of Fuzzy Sets and Systems was hatched by Enrique Ruspini, Madan Gupta and myself (Jim Bezdek) on a plane trip returning to the USA from Acapulco, Mexico in December, 1980. We had all given talks about fuzzy models at the *International Congress for Systems and Cybernetics*, and the lure of having our own forum for technical interchange of ideas in the field was a powerful one.

Lotfi Zadeh felt our enthusiasm for this venture was premature. We turned to King Sun Fu, who had been very supportive of fuzzy sets work, and he liked the idea. Fu was Bill Wee's thesis advisor. Wee wrote the first Ph.D. thesis on fuzzy pattern recognition just two years after Lotfi published his seminal paper in 1965.

> Wee, W.G. (1967). On Generalization of Adaptive Algorithms and Application of the Fuzzy Sets Concept to Pattern Classification. Ph.D. Dissertation, Purdue University, West Lafayette.

King Sun Fu felt the time was right for the formation of a professional society, so he agreed to chair a *North American Fuzzy Information Processing* (NAFIP) working group to formulate a plan for it. Subsequently he agreed to be the founding president of NAFIPS in 1981. The founding Board of Directors consisted of Jim Bezdek, Enrique Ruspini, Piero Bonissone, Richard Tong, Lotfi Zadeh, K. S. Fu, and Ron Yager. The Board did not meet physically during our first year of operation, but we did conduct a lot of business by *surface mail*. We initially called this organization a group. We had the idea that it should be called a society, but we did not decide to add the "S," for "Society" until 1985, the year that Enrique Ruspini filed the documentation for us to become a tax-exempt California corporation.

II. NAFIPS

I volunteered to organize the first conference under the banner of the NAFIP group. The conference, named NAFIP-1, was held on the campus of *Utah State University* (USU) in Logan, Utah, where I was an associate professor of Mathematics. The first physical meeting of the board of directors was held at my home in Richmond, Utah. A photograph of that event, Figure 1 below, was taken on the front porch of my house (Tanya, my 9-year daughter at the time, is holding King Sun Fu's hand). At this meeting, the board elected me to be the second president of NAFIP.

Figure 1. First NAFIP Board of Governors. Front: Left to Right: Paul Wang, Abe Mamdani, Tanya Bezdek, King-Sun Fu, Jim Yao, Lorenza Saitta. **Middle**: Janet Esfathiou, Richard Tong, Ronald Yager. **Back**: Marc Roubens, Philippe Smets, Piero Bonissone, Jim Bezdek, Enrique Ruspini, Elie Sanchez.

There were no papers submitted or published for this meeting; rather, it was a meeting of talks, which at that time was the main point of almost all professional conferences. I made up a book of abstracts that summarized the talks (my recollection is that there were 42 of them, ±3 or so). The talks were presented in classrooms at USU, mostly on blackboards with white chalk. I recall that Ron Yager, sitting in the front row at one of these talks, lit up a big cigar and happily puffed away during the talk. How times have changed!

I had the idea that this was an historic occasion, it being the first conference of the first professional organization dedicated to fuzzy sets and systems, so I made up a poster (on my first Apple Macintosh, with a huge 64kb of RAM!) for the meeting bearing the term "commemorative issue" at the bottom. I then had a print shop in Logan run off 75 copies of the poster, and each participant was given one at registration. I ended up with one clean unused copy of that poster, along with the signature version discussed below and displayed as Figure 2.

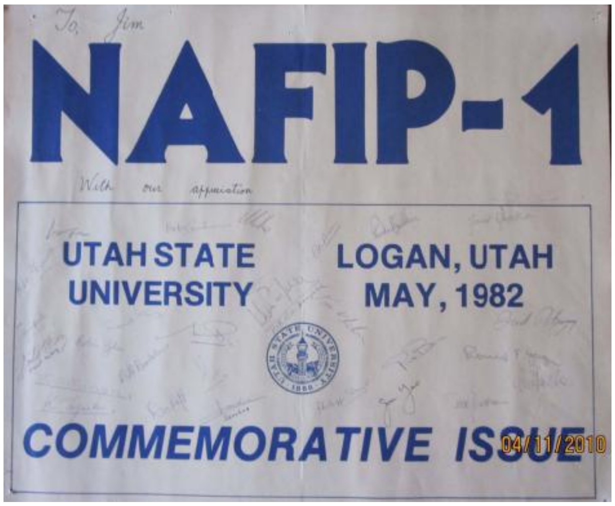

Figure 2. The signature poster presented to me

The fashion of the day was to have a glorious banquet, so of course we had one at the end of the meeting. Most of us (not quite all of us) drove up highway 89 towards Salt Lake City to a restaurant (I cannot recall its name) that was situated all by itself, nestled in the range of mountains the locals called the "Wellsvilles."

I had invited Brian Gaines to be our banquet speaker, but he had still not appeared when we left for the restaurant, so my anxiety about his participation in this event was palpable. Just as we were being served dessert, Brian arrived, having driven a rental car directly from the Salt Lake City airport to the restaurant. Brian blew into the room quite breathlessly, looking as disheveled as is possible for a distinguished practitioner of the fuzzy arts, gulped down a huge drink, went to the head of the room, and proceeded to give one of the funniest and best banquet talks you will ever hear. Here is Brian's account of that talk, sent to me via email on July 2, 2017:

> *Yes, I gave both the Logan and Kauai banquet speeches. How I became a banquet speaker is a mystery, particularly to me. The Logan one is very memorable. A few days prior to it I had been in Japan consulting with Toppan Printing and was scheduled to fly from Tokyo to Logan to arrive the day before I was due to speak. However the President of Toppan had to give a talk in English and asked me to stay an extra day and help him get it right. I did but then arrived at SLC the afternoon of the banquet, found I had missed the flight to Logan, hired a car and arrived just as dinner ended and I was due to speak. I never prepared such speeches—they were always extemporary, and of what I said I have no recollection.*

While Brian was talking, unbeknownst to me, the people at the banquet were quietly circulating a copy of the poster, which many of them signed (30 signatures I think), and King Sun Fu presented it to me at the end of the evening. It's a miracle that it survived even that night, because I was pretty well into my cups, and didn't remember much about that evening the next day. But it did survive, and I carried it around with me until I surrendered it to Rudi Seising for his collection of artifacts about the history of fuzzy sets. The copy of the poster shown here was scanned on April 11, 2010. I believe that Rudi has it to this day.

Some of the signatures are quite faded because of the ink used, some I don't recognize, some I can't read, and I am not sure that restoration is possible. Here is a list of the signatures that I can read, beginning in the upper left corner with King Sun Fu, and (roughly) traversing the poster clockwise and inwards.

> *K. S. Fu, Bob Gunderson, Colin Brown, Brian Gaines, Janet Esfathiou, Fred Petry, Ron Yager, Tove Jacobsen, Jim Yao, Piero Bonissone, Philippe Smets, Elie Sanchez, Don Kraft, Bill Buckles, Robin Giles, Paul Wang, Richard Tong, Lotfi Zadeh, Abe Mamdani, Kofi Dompere, Tom Whalen.*

I can name a few other attendees, whose signatures may or may not be on this poster: Mike Windham, LaDawn Haws, Lorenza Saitta, Rajesh Dave, Marc Roubens, Enrique Ruspini, Abe Kandel.

Sadly, King Sun Fu died unexpectedly while I was president of NAFIP. To recognize his importance to fuzzy sets and to our fledgling society, I designed a certificate of merit, and, with the permission of his wife, named it the King Sun Fu award. I believe that this award is still given by NAFIPS, and still uses my original design.

III. IFSA

Many things followed from our initial foray into the world of professional societies. For example, the Europeans in Logan decided that they wanted another society not specifically tied to North America. And so, they formed a working group towards this end which met at a conference in Kauai that I chaired in 1984. Figure 3 shows the poster I made for this event

Figure 3. The poster I made for Kauai

This working group led to the establishment of the *International Fuzzy Systems Association* (IFSA), who held their inaugural conference in Palma de Mallorca in 1985. Hans Zimmerman emerged from this group as the founding president of IFSA. I was subsequently elected as the second president of that organization.

IV. The IEEE CIS

This is how our fuzzy gang finally penetrated the IEEE. I was wandering around in the exhibits hall at the 1990 Int'l. Conf. on Fuzzy Logic and Neural Networks in Iizuka with Bernie Widrow (Stanford University). I had known Bernie for many years as he was a very close friend of my co-thesis advisor (Dave Block, Cornell University).

We saw many early examples of fuzzy logic being used in Japanese devices. There is a famous picture of Lotfi using the fuzzy vacuum cleaner there. We were looking at a washing machine that had a "smart fuzzy logic unit" to sense characteristics of the load and adjust the settings of the machine accordingly. I always thought the real cleverness in these devices was the sensor technology, but I never said that to Bernie.

Bernie looked at me and said "where are the Neural Network washing machines?" I said "there won't be any Bernie, they belong in a different neighborhood." He said "well, I intend to go home and tell the IEEE that we are at (economic) war with Japan, and we need to get into the fuzzy logic business." Not three months later, I got a call from Bob Marks, the president of the *IEEE Neural Networks Council* (NNC). He asked me to join the NNC and help them "get into fuzzy logic." And so, I did.

A number of things happened for our community in the next few years. In 1992, I chose a name that has stuck for the first *IEEE International Conference on Fuzzy Systems, viz.* "FUZZ-IEEE," and chaired the conference, held in San Diego, CA. In the same year I chaired a committee to propose an IEEE journal devoted to fuzzy logic, and subsequently became the founding editor of the *IEEE Transactions on Fuzzy Systems*. And finally in 1992, I published a paper defining what I meant by the term "*computational intelligence* (CI)."

> Bezdek, J. C. (1992). On the Relationship between Neural Networks, Pattern Recognition, and Intelligence, *Int. J. Approximate Reasoning*, 6(2), 85-107.

My aim was to differentiate activities based on pattern recognition (so-called artificial neural networks was one of them) from what was called *artificial intelligence* (AI) at that time. As used then, AI meant rule-based expert systems, whereas now, the term seems to mean "deep learning" using giant neural networks!

In any case, I suggested to the NNC Adcom that the term CI would be a good one to describe our scope when we evolved from an IEEE council to a society. Well, that did happen, and then I extended my suggestion to include using it for our first world congress, held in Orlando in 1994. Hence the terms CIS and WCCI. If you are interested in this aspect of the CIS, you can read a complete account of it in:

> Bezdek, J. C. (2016). Computational Intelligence: What's in a Name?, *IEEE SMC Magazine*, 2(2), 4-14.

I will finish this note by recognizing the contributions of several other people who made life easier for many of us in the early days.

V. Unsung Heroes of the Early Days

Lotfi was of course at the forefront of everything that happened. I have also discussed the important roles played by King Sun Fu and Bernie Widrow. There were several other people who came to our rescue when the forces of darkness were hurled against us. I would like to mention three of them here. *Forces of darkness*? Yes, there were any number of people who, for one reason or another, were quite aggressive and confrontational towards the fuzzy community (well, this hasn't changed much since then, has it?). See these two papers if you find this interesting.

> Bezdek, J. C. (1994). Fuzziness vs. Probability - Again (!?), *IEEE Trans. Fuzzy Systems*, 2(1), 1-3.

> Bezdek, J. C. (2013). The parable of Zoltan, in *On Fuzziness*? A Homage to Lotfi A. Zadeh, eds, Seising, R., Trillas, E., Termini. S. and Moraga, C. (eds.): Springer-Verlag (Studies in Fuzziness and Soft Computing, vol. 298), 1, 39-46, 2013.

Azriel Rosenfeld (University of Maryland) wrote many papers about fuzzy graph theory and related topics in the early ‘70s. He had several students working in the area. He was always ready to answer questions, and never wavered in his belief that our embryonic field was an important one.

Theo Pavlidis (SUNY Stony Brook) was the editor of IEEE TPAMI from 1982-1986. He encouraged submissions to TPAMI while other journals were actively discouraging work in our field. He was personally responsible for chaperoning some of the earliest papers on fuzzy topics through the gates of TPAMI.

Andrew Sage (University of Virginia) was instrumental in the creation of the IEEE SMC society, and served as editor the IEEE TSMC from 1972-1998. Like Theo Pavilidis, Andy had an open mind about our field, and gave us many opportunities that seemed dear to us at the time. And of course, the SMC was the mother ship for the eventual launch of our own IEEE CIS.

And the rest, as they say, is history

The Fuzzy Moment That Changed My Life

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Abstract—This is a short personal story of how I got involved in fuzzy set related research.

Keywords—fuzzy pattern recognition, Jim Bezdek, Lotfi Zadeh

MY STORY

My academic training, back in the 1970s was in mathematics. Both my Masters Thesis and PhD Dissertation were as pure as the new fallen snow. However, during my schooling I developed an appreciation, in fact a love, for practical applications of math principles through these new-fangled computers. I took some courses and was very fortunate to get a student research position in a very interdisciplinary computational research program housed in the Electrical Engineering Department at the University of Missouri (MU). Computer Tomography (CT) was a hot topic in the mid-1970s, and my first engineering project was to build CT reconstruction algorithms for nondestructive testing via neutron beams at the Missouri University Research Reactor (MURR). Really cool mathematics (Radon Transforms) and super cool FORTRAN programs 😉. That's where I learned about and fell in love with image processing.

My first "images" were numeric outputs on 132 column printer paper, spread out and taped together on the lab floor. I segmented these images with a marking pen and we looked at them from a distance – now it kinda gives me a Salvador Dali "Abraham Lincoln" vision. General Electric Medical Systems Division invested heavily in both head and full body CT scanners and were looking for ways to enhance the desirability of purchasing these expensive machines, i.e., searching for additional off-line applications of the CT imagery. Our program at MU received a research contract from GE to develop a radiation therapy treatment planning system that used CT and nascent computer graphics to improve the estimation and visualization of radiation dosage to the body. I was able to develop some of the early image processing techniques on stacks of CT slices, probably the first to display coronal and sagittal representations overlaid with graphical visualizations of radiation dose distributions. The hook with CT was that it allowed us to accurately determine the contours of the body and internal organs/structures to correct radiation dosage distribution calculations. This stand-alone CT-based Radiation Treatment Planning system was built on a Data General Eclipse minicomputer with 64K bytes of memory (yes, that a "K", not M, G, or T) and a whopping 300MB disk drive that was the size of a washing machine and a 9 track magnetic tape drive as image input. Heady stuff.

I was then (and still am) enamored with pattern recognition and its applications. I mean, who doesn't love Bayes Rule and the perceptron algorithm? So, what does this have to do with fuzzy sets? I was a busy Assistant Professor using pattern recognition, image processing and computer graphics working mostly on applications involving digital medicine, food science, and laboratory automation. But, I hadn't really found my professional community. I tried (maybe not hard enough) to integrate into the SIGGRAPH gang, but without much success. Then in the early 1980s a graduate student brought in a book he found in the library (yes, we actually went to the library and wandered through the stacks looking for those new or hidden gems). This is the text that changed my life:

> James C. Bezdek, *Pattern Recognition with Fuzzy Objective Function Algorithms*, Plenum Press, New York, 1981.

I "knew" about fuzzy sets in a "mathy" kind of way. There was a fellow grad student in Math who was proving theorems about fuzzy topology – so that was my mind set. This book was different; it addressed my passion. We spent the summer reading and studying, and a new world opened up for me: fuzzy partitions (actually defined by Enrique Ruspini), the Fuzzy C-Means theory, algorithm and applications, and, really important to me at the time, ideas about using soft labels in classifier design. Of course, all fuzzy set research stems from the Lotfi Zadeh root. I studied everything I could get my hands on. I was particularly inspired by the massive 3 part series of papers by Lotfi on linguistic variables that helped me build solid mathematical foundations for much of what followed. It's hard to state just what an immense influence Lotfi Zadeh had on us all. I remember early on Ron Yager saying to me something to the effect that Lotfi was an implicit co-author on every paper we wrote.

However, it was the Bezdek book set me on the path to explore connections between fuzzy sets and pattern recognition/image processing. I had a flurry of fuzzy classifier ideas in the mid to late1980s, all rooted in the basic message of that 1981 Bez book. A few of my favorites are the fuzzy K-NN, early explorations on fuzzy integral for pattern recognition and image segmentation, fuzzification of the perceptron algorithm, and fuzzy rule bases for object recognition. It wasn't until the early 1990s that I really got into fuzzy clustering.

But I'm digressing, as usual. Back to the main story. I sent my first fuzzy pattern recognition papers to the IEEE Applied

Imagery Pattern Recognition Workshop (AIPR) featuring what we'd now call type 1 fuzzy weighted averaging. I read a small monograph on using fuzzy sets to model words and to assess risk by doing fuzzy set based weighted averaging of the factors. The papers looked at simple examples of temporal and multispectral fusion:

> Keller, J., Nafarieh, A., Wootton, J., and Hobson, G., "Fuzzy Confidence Measures in Multitemporal Imagery," IEEE Applied Imagery Pattern Recognition Workshop, Baltimore, MD, October 1985.

> Wootton, J., Hobson, G., Luetkemeyer, K., and Keller, J., "The Use of Fuzzy Set Theory to Build Confidence Measures in Multisensor Imagery," IEEE Applied Imagery Pattern Recognition Workshop, Baltimore, MD, October 1985.

I nervously presented, was puzzled by a question about statistical methods, and thankfully went out to the coffee break. I had no formal training in fuzzy set theory and had no idea that there was any passionate controversy about fuzzy uncertainty. A guy came up and yelled at me about how I should be using a statistically optimal classifier. **Yikes!** I stammered a reply something like "That would be great, but I have no statistics to build it with". The real reason is deeper, but that's what I came up with at the time. Anyway, that night was the banquet, and they had an open bar and I was a young Assistant Professor … I was standing after the meal and heard "Keller!" from behind. Worried that my confronter had returned, I spun around to see this guy with a beard, a baseball cap, and a flannel shirt sticking out his hand and saying "Jim Bezdek, here". I was stunned and all I could manage was "I read your book". Bez told me some things I won't repeat here, and that I needed to come to the next NAFIPS meeting to meet many of the fuzzy community (at least from North America). At that 1986 meeting in the French Quarter, I was welcomed and embraced by that group of fuzzy researchers, led by Professor Zadeh - I found my home. Hats off to Jim Bezdek and Pattern Recognition with Fuzzy Objective Function Algorithms.

The rest, as they say, is History.

Figure 1. Jim with two of his fuzzy inspirations in 2005.

On a 1965 Great Idea

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***Abstract*—A personal view of the author about Fuzzy Sets on the occasion of the 60 anniversary of its appearance, including his role in its development (particularly in Spain), his contributions to the topic, and future prospects.**

I.

Sixty years ago, in 1965, Lotfi A. Zadeh (1921-2017) published his famous paper ‘Fuzzy Sets’ in the journal ‘Information and Control’. It was a paper that immediately attracted the attention of many people for alternatively either praising and also worshiping, or denigrating, the ideas presented in it. Trying to concede scientific relevance to imprecision seemed, to some scientists and in that epoch, a waste of time and, also, an extravagance, and, even worse, a sample of wistful thinking. Zadeh ended up being insulted as a researcher.

Such attitudes continued up to when Japanese industries did place in the markets what were known as ‘Fuzzy Logic Products’ thanks, in an important part, to the work of Professor Michio Sugeno (1940-2024), then at the Tokyo Institute of Technology, among a pleiad of excellent Japanese researchers. Such products meant the technological success of Fuzzy Sets and almost reduced contrary voices to silence.

If the industrial relevance of Zadeh’s new ideas did show the inadmissibility of the critics, anyway such voices still lasted, more or less sotto voce, for some time. I remember well, for instance, the words of a brilliant American Mathematician who, in the last seventies of the former 20th Century, said to me (or perhaps better, ‘launched upon me’): ‘Did Zadeh pretend arrange the world just with functions into the unit interval?’; he was criticizing Zadeh without understanding what he was actually doing. Even the famous mathematician Karl Menger (1902-1985), who did show respect for Zadeh’s idea, said to me that his own, but probabilistic, ‘Hazzy Sets’, were most interesting and useful than Fuzzy Sets, when they are but a particular case and much of linguistic vagueness is not probabilistic in nature as, years ago, did show Max Black (1909-1988) with his ‘Profile Functions’ with which he tried to represent words and meant a true antecedent of Zadeh’s Membership Functions.

In fact, American Critics did not capture well Zadeh’s ideas, and, actually, were some brave engineers out of the United States, like Professor Abe H. Mamdani (1942 - 2010), then at the London’s ‘Queen Mary College’, who applied fuzzy sets to control mechanisms. Today, Zadeh’s 1965 paper is one of the most cited papers in the last sixty years.

II.

In Spain, like in many countries, those that knew those ideas were in the praise side, and the first among them were professors Alfredo Deaño (1944-1978) in the ‘Complutense University of Madrid’, Francisco Azorín (1914-1989) in the ‘National Institute of Statistics’, and the author of this paper who, then in the ‘Polytechnic University of Barcelona’, knew fuzzy sets in the Summer of 1974, after reading an interview with Professor Arnold Kaufmann (1911-1994) in a French newspaper and concerning his then recent book on the ‘sous ensembles flous’.

I had the idea and the chance of attracting some researchers, then very young people, to study and develop Fuzzy Sets Theory and generating, jointly and finally, one of the world’s best communities on the subject. Deaño just wrote a nice informative chapter on Fuzzy Logic in his 1975 book ‘Introducción a la lógica formal’, and Azorín tried to use Fuzzy Sets for imprecise questions/answers in Statistical Surveys, a subject on which he published some papers. By my part and from 1977 up to now, I published more than four hundred papers, twelve books, supervised more than twenty doctoral dissertations on Fuzzy Logic subjects being mine the first book in Spanish on Fuzzy Sets [E.Trillas, 1979.’Conjuntos Borrosos’.Ed. Vicens-Vives, Barcelona]. Our Spanish community arrived, thanks to the intellectual drive of people like Professor Miguel Delgado in the ‘University of Granada’ to reach, at the beginning of the 21st Century, the third world’s place by number of publications in Fuzzy Logic in a country that then, and by its total number of scientific publications, was in between the twelve and ten world’s position.

III.

I met Lotfi Aliasker Zadeh in Barcelona in July 1977; it was in occasion of the ‘First World Conference on Mathematics and the Service of Man’ of which I was one of its organizers and invited Zadeh to deliver a Plenary Lecture whose subject, a view of Fuzzy Sets from a new Theory of Possibility meant, for me and jointly with the concept of a Fuzzy Entropy introduced in 1972 by Aldo de Luca (1941-2018) and Settimo Termini (1945), to definitively falling in love with Zadeh’s ideas, as well as the starting point of a deep and unforgettable friendship with Zadeh, that lasted for forty years up to his death in 2017. I must openly declare how much I miss him.

Zadeh was for me both a friend and, mainly, a true teacher who not only gift us with Fuzzy Sets and Fuzzy Logic as the central part of his scientific inheritance, but also with what he called ‘Computing with Words’ (CwW), and ‘Soft

Computing' (SC), introduced by him in the mid nineties of last 20th Century. From these days it already did pass a period of time sufficient to expect some substantial progress on CwW that, nevertheless, only seems to be scarcely reached, let me say, in just minor questions.

For my part, and with the aim of contributing even indirectly to such progress, I tried to reconsider the same concept of a Fuzzy Set, as well as what is in the back of Fuzzy Logic. Leaving aside why the basis of CwW did receive a so scarce attention, and with just the brief and painful comment that I wonder myself if, perhaps, the current fashion on Artificial Intelligence drains the attention of too many young researchers. Who knows the influence insistent news in the media can have in such fashion; nothing too different, notwithstanding, to what, long time ago, happened with Relativity Theory and Atomic Energy attracting students to Physics.

IV.

I will refer to just a part of what I did concerning the use of Fuzzy Logic to better understanding Common Sense Reasoning after devoting my efforts to study the validity of classical laws with fuzzy sets, a subject that produced me surprises like it was the validity with fuzzy sets of the (very crisp!) set's law of 'perfect repartition'. If sets are inspired in true collections of objects, fuzzy sets are but mind linguistic entities: if everybody can see either a set of watermelons, or a set of pencils, nobody can neither see, for instance, the fuzzy set of young Spaniards, nor that of tall Italians. Usually fuzzy sets are not sets.

In addition, and contrary to what, following Zadeh's own 1965 definition, is said in all treaties of Fuzzy Set Theory, given a linguistic label a membership function alone can't define a fuzzy set since, and as everyone using fuzzy sets knows, the same fuzzy set can be exhibited by several membership functions just preserving some, let me say, 'family resemblance form'. Confusing a fuzzy set with one of its membership functions is a conceptual mistake and, for my taste, the existing theory lacked a (formal) qualitative point of view. But it was the hint from the same Zadeh: 'fuzzy sets do concern the 'extensional meaning' of their linguistic label', that conducted me to the decisive word 'meaning' I immediately related with the famous Ludwig Wittgenstein (1889-1951) statement: 'The meaning of a word is its use in language', concerning a purely qualitative view and reflecting what is done in practice by designing membership functions.

Since linguistic labels are but predicates, that is, adjective words 'saying something' objects into consideration do show, I thought that 'use' can refer to variation, and 'extensionality' to degree, as well as that the ground idea is of a relational character. When variation is done under separate, identical and clearly distinguishable steps, degrees, the word is precise, with a rigid, crisp, meaning; when the steps are not distinguishable, the word is imprecise, with a vague, fuzzy, meaning.

All those reflections conducted me to define the qualitative or primary meaning in a universe of discourse, X, of a predicative adjective, P, by means of the binary and perceptive relation $< P$ in the universe X, defined by: '$x < Py \Leftrightarrow x$ is less P than y', that, if P names some property p of the elements, is equivalent to: 'x shows property p less than y shows it'. If it does not seem rare to accept that $< P$ is reflexive, only exceptionally it will be antisymmetric, and less exceptionally transitive. In this way, we arrive to a graph $(X, < P) = P$ reflecting the use of P in X; a graph that can or cannot have maximal or minimal elements and, called the 'qualitative or primary meaning' of P on X, can be also seen as the fuzzy set in X with linguistic label P. In addition, when given P in a universe X, and a graph P is known, it is said that P and p are 'measurable' on X and, if such graph is unknown, that P and p are 'metaphysical on X'.

Since it is clear that both Science and Technology mainly deal with measurable, non metaphysical properties it is, perhaps, that dealing with metaphysical concepts is what separates Science and Philosophy. It seems appropriate to pose the question: is it the 'thought's demarcation' some philosophers did search for?

Finally, and to see what is a membership function for P, it just suffices to introduce extensionality by means of some measure $mP : X \to [0, 1]$, that is, by a function whose basic properties are: 1) $x < Py \Rightarrow mP(x) < mP(y)$; 2) If u is a minimal in the graph, $mP(u) = 0$; 3) If v is a maximal in the graph, $mP(v) = 1$, a concept I defined by inspiring me on the Michio Sugeno definition of a (non necessarily additive) fuzzy measure in his 1974 Ph.D. Thesis,'Fuzzy Integrals and Its Applications', as well as in some hints appearing in the paper [R.Capocelli, A. de Luca, 1973, 'Fuzzy Sets and Decision Theory', 'Information and Control':23:446-473].

With all that we finally count with the scalar magnitude $(X, < P, mP) = Pm$, a quantity I called a quantitative or total meaning of P in X, and with which the new and clearly linear relation $< m$ on X, defined by: $x < my \Leftrightarrow m(x) < m(y)$, is reflexive, antisymmetric and transitive, a partial order, and contains the former relation $< P$ since: $x < Py \Rightarrow mP(x) < mP(y) \Leftrightarrow x < my$, and when there is coincidence between both relations $< P$ and $< m$ it is said that mP reflects meaning perfectly. It can be also said that measuring amplifies the primary meaning, an observation alerting practitioners (always working with a Pm instead of P), of only attributing semantically to P what comes from $< P$, since what $< m$ adds to the primary meaning of P is not qualitative information. In addition and obviously, the measure mP can be seen as a membership function of P since it clearly measures up to which extent each element x (in X) is P. It is also obvious that properties 1,2 and 3 of mP are not, in general, enough for specifying a single function mP: in general, to specify a measure more information on the behavior of P in X is necessary. Notice that magnitude Pm can be seen as the state in which P is currently manifested and, also, as a 'working fuzzy set' version of P.

V.

The four pages limitation makes impossible to extend the paper with the author's view of Commonsense Reasoning, where he did open a window toward clarifying what can be understood by 'inducing' after formalizing deducing, abducing, conjecturing or guessing, and refuting in the framework of a small set of axioms he called the 'skeleton' of reasoning allowing to prove as theorems the Aristotle's Principles of Non-Contradiction and Excluded Middle. He also introduced 'speculations', conclusions orthogonal to the premise that logicians did left aside, as well as considered the 'act of reasoning' as an inferential zigzag around the premise done by sequentially alternating deduction and abduction, and reinforcing the view of the (essential) role language plays at each step in the act of reasoning. These contributions can be studied in the books: 1) E.Trillas, S. Termini, M.E. Tabacchi, 2022, 'Reasoning and Language at Work / A Critical Essay'. Springer. 2) E. Trillas, 2022, 'La génesis de la lógica / Reflexiones ingenuas'. EUSC, translated into English as: 'The Genesis of Logic / Reflections on the Origins, Principles and Paths of Common-sense Reasoning', 2024, Springer, and also in the paper [E,Trillas and J,M,Terricabras, 'A Scrutiny on Representation / A New View on Meaning and Reasoning'; 'Archives on the Philosophy and History of Soft Computing' 1/2016:1-50].

VI.

Last but not least, let's come back to Zadeh's heritage by stating that without his intellectual support and help to all of us, the successes reached by the Spanish 'Fuzzy Community' would have been much less known around the world than they were; even, perhaps and possibly, remaining almost unknown. It should be added that Zadeh's ideas are still alive, and that the critical reading of his contributions is worth deserving for anyone wishing to make original contributions to Fuzzy Logic and, among them and especially, his (rather curious, surprising and full of hints) 2012 book 'Computing with words. Principal Concepts and Ideas', in Springer.

The flexibility fuzzy sets exhibit is essential to deal with Natural Language and Commonsense Reasoning, and it is more so once its closeness to the semantic concept of meaning is known as well as when the 'skeleton' shows that the rigid lattice structure is just a local one in language. Actually, Zadeh's ideas show a possible path towards a new and still unknown Natural Science of Reasoning and Language since, at the end, they are but Natural Phenomena like for Physics are Movement, Energy and Time. In addition, reinforcing the relationship between Fuzzy Logic, as a Mathematical Science of Meaning, and the Neurosciences, seems to be an enterprise of a relevant interest and whose success can be envisaged in the book [U.Sandler, L.Tsitolovsky, 2008; 'Neural Cell Behavior and Fuzzy Logic'. Springer']. I strongly recommend researchers to do any trial concerning to (creatively) establish such still unknown Natural Science, as well as to look for its relationship with Brain Studies; a recommendation I also extend to beginners since it is a challenging scientific opportunity for arriving to an Arquimedian Eureka!

Sixty Years and Still Going Strong Fuzziness in AI's Near Future

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***Abstract*—Fuzzy sets, FL, CWW and CWP have repeatedly led to successful extensions in theory and applications in the history of artificial intelligence (AI). Even in today's world, where AI no longer stands only for symbol-processing, but sub-symbolic systems in many circles and machine deep learning dominates this field, FL, CWW and CWP have still a word to say, and will lead to interesting variants; the fusion of quantum logic and AI is a possible example of how FL again offers space for new developments in AI.**

***Index Terms*—Fuzzy set theory, LLMs, Fuzziness.**

I. FL, CWW and AI

From the very beginning, fuzzy sets were associated with an application potential that Lotfi Zadeh, their founder [1], had suspected in the early years in the humanities and social sciences [2]. He was surprised when, in the early 1970s, a revolution in the field of control theory and technology [3], initiated by Sedrak Assilian and Ebrahim Mamdani, set in motion the first great success story of fuzzy sets [4], [5]. When computers became much faster, so that the control of complex systems, even with classical controllers, became just as successful as fuzzy control, the sensational applications of fuzzy sets shifted to clustering, data mining and information mining. In the 1950s, the field of artificial intelligence (AI) had become a field of research to build computers and computer programs that act 'intelligently'. AI methods were methods to compute with numbers and find exact solutions. However, not all problems could be solved with these methods. On the other hand, humans are able to resolve such tasks very well, as Zadeh mentioned in many speeches and articles over the last decades. In conclusion, he stated that "thinking machines" do not think as humans do. From the mid-1980s he focused on "Making Computers Think like People" [3]. For this purpose, the machine's ability "to compute with numbers" should be supplemented by an additional ability that is similar to human thinking: Computing with Words (CWW) and Perceptions (CWP).

In the mid-1980s, the philosopher John Haugeland (1945-2010) coined the term GOFAI (Good Old Fashioned Artificial Intelligence). He distinguished it from newer AI approaches that used artificial neural networks and classification trees [6]. GOFAI was based on the assumption that aspects of intelligence can be achieved by manipulating symbols in a machine. can be achieved. To this end, algorithms were formulated in programming language and then executed command by command.

These NFAI ("New Fashioned AI", or "New-Fangled AI") did not suddenly replace GOFAI, rather the developments of both AI approaches overlapped and they are both still being pursued. NFAI systems as Artificial neural networks (ANN), evolutionary algorithms, genetic algorithms,ant-algoithms and statistical algorithms as classification trees (CT) are algorithms that search for patterns in data sets. In this way, they accomplish something that humans can do poorly or not at all, namely mastering large amounts of data. With these and now many other data-driven algorithms, machine learning (ML) can perform tasks that humans are not capable of.

In 1990, Lotfi Zadeh coined the label 'Soft Computing' (SC) to name an interdisciplinary field that covers different approaches to AI that had been developed but weren't part of the mainstream of AI:

> By design, soft computing is pluralistic in nature in the sense that it is a coalition of methodologies which are drawn together by a quest for accommodation with the pervasive imprecision of the real world. At this juncture, the principal members of the coalition are fuzzy logic, neurocomputing, evolutionary computing, probabilistic computing, chaotic computing, and machine learning. [7]

When Hans-Jürgen Zimmermann, founding editor of the journal *Fuzzy Sets and Systems*, foresaw that the development of 'hybrid systems' of 'fuzzy neuro-evo-combinations' would continue in the future, he deliberated about a name for the common field of research, which would then also become the subtitle of the journal. These concepts seemed to be attractive in different ways and also varied with respect to their expressive power. He suggested calling the field 'soft

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computing and intelligence' since the other concepts seemed to place too much emphasis on 'computing' 'which is certainly not appropriate, at least for certain areas of fuzzy set theory.' [8]. In recent decades, the term AI has increasingly become synonymous with ML and today large language models (LLMs) also offer a huge field of application for fuzzy sets.

Machine learning models enable predictions of future events based on measurements and data generation and large language models (LLMs) predict which word matches existing words based on previous word classifications. Fuzzy sets can be useful for both the measurement and the matching, as well as for the adaptation of these predictions, known as "learning".

A specially useful concept that can translate well from the realm of Fuzzy Logic (FL) to AI systems is that of Computing with Words (CWW, also CWP when perceptions are included) [9]. This is the idea that the late father of Fuzzy Logic, L. A. Zadeh, has envisioned in the last years of his long career as the unifying concept behind all of Fuzzy Logic (FL): a way to eschew for good the necessity of mediating reasoning through numbers, and to achieve a better, more ecological way of reasoning:

> Fuzzy logic has come of age. Its foundations have become firmer, its applications have grown in number and variety, and its influence within the basic sciences – especially in mathematical and physical sciences – has become more visible and more substantive. Yet, there are two questions that are still frequently raised: a) what is fuzzy logic and b) what can be done with fuzzy logic that cannot be done equally well with other methodologies. [10, p.77]

The basic idea is that the reducing all our reasoning to numbers, a necessity that comes out of the need to represent concepts in computations and that is a mainstay of classic, control-bound FL, should be kept at a lower level, and that the higher level of representation is the one the effort should be directed to. In this idea there is a rekindling of FL with classical logic, which is in effect a way of computing with words. Clearly examples of that are Propositional and Predicate logics, where the centre of computation are terms that represent reality, and computation itself is an aggregation of such terms following a number of simple rules. CWW aims at expanding the same idea to include intrinsic properties of the human language such as imprecision and vagueness (as the original FL expands boolean algebra):

> There are two major imperatives for computing with words. First, computing with words is a necessity when the available information is too imprecise to justify the use of numbers; and second, when there is a tolerance for imprecision which can be exploited to achieve tractability, robustness, low solution cost and better rapport with reality. Exploitation of the tolerance for imprecision is an issue of central importance in CW. At this juncture, the computational theory of perceptions — which is based on CW — is in its initial stages of development. In time it may come to play an important role in the conception, design and utilization of information/intelligent systems. [. . .] The role model for CW [. . .] is the human mind. [9, p.103]

While the original concept has not (yet) lived up to its promises, it is possible to find traces of the original ideas in many of the more recent implementations of AI – which is, in our opinion, unavoidable: it really seems impossible to build, e.g., a credible model of human language without keeping in account the inherent properties of it, and without confining numbers at a lower, less accessible level. Just as a hint, two examples are given.

LLMs

LLMs, the algorithm behind most of the current generative language systems in AI, are based on tranformers, which in turn rely on the concepts of Embedding and Attention [11].

Embeddings are internal representations of words or tokens that allow a language model to understand and work with language more effectively. When text is input into a large language model, each word or token is transformed into a format that captures its meaning and how it relates to other words. These representations are not fixed; they evolve based on how words are used in context, allowing the model to distinguish, for example, between different senses of the same word.

These representations also include information about the position of each word in the sentence, helping the model to to grasp the structure and flow of language. As the model processes input through multiple layers, these internal representations become increasingly refined, enabling the model to generate coherent responses, understand intent, and perform a wide range of language-related tasks.

It is as well possible to compute using embeddings, as the representation of words is subject to operations which are akin to the application of operators and words to words, to obtain meanings that are slightly different depending on the role of the word in a phrase. This mechanism is a clear example of CWW, as it uses words as the main representation of concepts and calculation to elaborate. While such meanings are (obviously) stored as numbers (or, more precisely, vectors), such representation is removed from the human experience of an LLM.

QNLP

Quantum Natural Language Processing (QNLP) us a recent advancement of Quantum Logic aimed at representing Languages from the point of view of Quantum Computing. In a recent paper, Coecke et al. [12, p.1] have provided

> conceptual and mathematical foundations for near-term quantum natural language processing (QNLP). the quantum model for natural language that we employ canonically combines linguistic meanings with rich linguistic structure, most notably grammar. In particular, the fact that it takes a quantum-like

> model to combine meaning and structure establishes QNLP as quantum-native, on par with simulation of quantum systems. (...)

The system works through diagrammatic reasoning:

> Firstly, the quantum model interprets language as quantum processes via the diagrammatic formalism of categorical quantum mechanics. Secondly, these diagrams are made via the ZX-calculus translated into quantum circuits.

The paper details as an example the construction of a verb that connects two entities by superposing all the entity pairs that satisfy the relationship that represent the verb. A computation of the inner product of the two entities gives a value that represents the intensity of the relationship between the two entities.

This mechanism gives results that are essentially equivalent to a fuzzy degree, and while (again) numbers are used as a vehicle of computation, the entire system is essentially based on concepts and combinations of such concepts through the use of quantum computation.

II. Conclusions

Over the course of artificial intelligence (AI) development, the theoretical constructs and applied methodologies associated with fuzzy sets, fuzzy logic (FL), computing with words (CWW), and computing with perceptions (CWP) have consistently played a pivotal role in driving progress. These concepts have not only enhanced the expressive power of AI systems but also provided robust frameworks for modeling imprecise, uncertain, and linguistically nuanced information—features that are often inherent in real-world problems and human cognition.

Despite the current dominance of sub-symbolic approaches—particularly those based on machine learning and deep neural architectures, which have gained widespread attention due to their impressive empirical successes—the foundational principles of FL, CWW, and CWP continue to offer critical value. In fact, these paradigms remain highly relevant in addressing challenges related to interpretability, explainability, and human-aligned reasoning, which are increasingly recognized as limitations of purely data-driven systems.

Today, FL, CWW, and CWP persist as vibrant areas of inquiry, with the potential to inspire new theoretical extensions and innovative applications. One particularly promising direction lies in the convergence of fuzzy logic with emerging computational paradigms, such as quantum computing and quantum logic. This intersection presents fertile ground for the creation of hybrid AI models capable of integrating probabilistic reasoning, linguistic uncertainty, and quantum indeterminacy in a coherent and meaningful manner.

In this context, fuzzy logic does not merely retain a historical or auxiliary role in AI; rather, it positions itself as a flexible and adaptive formalism that continues to enrich and expand the field. Its capacity to serve as a bridge between symbolic, sub-symbolic, and quantum paradigms suggests that FL and its associated methodologies are well-equipped to contribute to the next generation of intelligent systems.

References

[1] L. A. Zadeh, “Fuzzy sets,” *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] ——, “Towards a theory of fuzzy systems,” in *Aspects of Network and System Theory*, R. Kalman and N. DeClaris, Eds. Holt, Rinehart, and Winston, New York, 1971, pp. 469–490.

[3] ——, “Making computers think like people,” *IEEE Spectrum*, vol. 8, pp. 26–32, 1984.

[4] S. Assilian, “Artificial intelligence in the control of real dynamic systems,” *Ph.D. Thesis*, vol. No. DX193553,, 1974.

[5] E. M. Mamdani and S. Assilian, “An experiment in linguistic synthesis with a fuzzy logic controller.” *International Journal of Man-Machine Studies*, vol. 7, no. 1, pp. 1–13, 1975.

[6] J. Haugeland, *Artificial Intelligence: The Very Idea*. Cambridge, Mass.: MIT Press., 1985.

[7] L. A. Zadeh, “Foreword,” *Applied Soft Computing*, vol. 1, no. 1, pp. 1–2, 2001.

[8] H. J. Zimmermann, “Editorial,” *Fuzzy Sets and Systems*, vol. 69, no. 1, pp. 1–2, 1995.

[9] L. A. Zadeh, “Fuzzy logic = computing with words,” *IEEE Transactions on Fuzzy Systems*, vol. 4, no. 2, pp. 103–111, 1996.

[10] ——, “Fuzzy logic, neural networks, and soft computing,” *Communications of the ACM*, vol. 37, no. 3, pp. 77–84, 1994.

[11] A. Vaswani, N. Shazeer, N. Parmar, J. Uszkoreit, L. Jones, A. N. Gomez, Łukasz Kaiser, and I. Polosukhin, “Attention is all you need,” in *Advances in Neural Information Processing Systems (NeurIPS)*. Curran Associates, Inc., 2017, pp. 5998–6008. [Online]. Available: https://arxiv.org/abs/1706.03762

[12] B. Coecke, G. de Felice, K. Meichanetzidis, and A. Toumi, “Foundations for near-term quantum natural language processing,” *arXiv preprint arXiv:2012.03755*, 2020. [Online]. Available: https://arxiv.org/abs/2012.03755

60 Years of Fuzzy – Past, Present, and Future: A Personal Opinion

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***Abstract*—In this paper, we express our view of what were the main challenges that faced fuzzy activities, how some of these challenges were successfully resolved and what is, in our opinion, the main remaining challenges.**

***Index Terms*—Fuzzy techniques, fuzzy data processing, interval computations, machine learning**

I. The Birth of Fuzzy: A Brief Description of the Main Motivation and Main Ideas

Why fuzzy: an important practical challenge that led to fuzzy. In the early 1960s, Lotfi Zadeh, who was at that time one of the world's best specialists in control and a co-author of the most popular control textbook, noticed that in many practical cases, there appeared as unexpected limit on what automatic control can achieve. In many cases, even with the optimal algorithms, the automatic control was not performing as effectively as the manual control by skilled controllers.

Of course, one cannot do better than the optimal control, so the problem was that the models of the controlled systems – models used to design automatic control – were not perfect: skilled controllers knew something about these systems that was not incorporated into these models. In many cases, these controllers explicitly explained what exactly was missing – but the problem was that they could only explain their knowledge by using imprecise ("fuzzy") words from natural language, like "small". We understand these words, but automatic systems need exact instructions.

Many of these expert controllers were skilled in mathematical techniques, but they still could not describe their knowledge in precise terms. This inability is normal. For example, most people can walk and run, but can we explain in precise terms how we do it? It is doubtful.

How Zadeh encountered this challenge. Since natural language is the only way this additional knowledge is available, Zadeh decided to translate such imprecise knowledge into precise computer-understandable terms. He also coined the word "fuzzy" to describe techniques for such translation, and he proposed specific techniques for this purpose.

One of Zadeh's main ideas came from the fact that a "crisp" (= precise) property like "positive" can be describing by assigning, to each possible value x of the corresponding quantity, the truth value "true" or "false" depending on whether x has this property (e.g., whether x is positive). In the computer, "true" is usually represented as 1 and "false" as 0, so we assign, to every x the value 0 or 1.

We cannot do that with properties like "small", since for some values x, we are not 100% sure that this value is small, and we are not 100% sure that the value is *not* small. Such values are small *to some extent*. So Zadeh proposed to ask the experts to estimate, for each x, the extent to which the corresponding property is satisfied by a number.

Such an estimation is not a new idea: we do it every time we fill a survey on how happy we are with some service or some purchase. Students do it in their annual evaluation of instructors, etc.

In practice, we can have many different scales. To make them uniform – and to get closer to the crisp 0-or-1 case, Zadeh suggested to reduce all the degrees to the scale from 0 to 1.

Thus, according to Zadeh's original idea, to describe each imprecise property, we assign, to each possible value x of the corresponding quantity, a value $m(x)$ from the interval $[0, 1]$ describing to what extent x satisfied this property. Zadeh called the resulting function $m(x)$ a *membership function* or, alternatively, a *fuzzy set*.

Steps of the resulting methodology. The above idea naturally leads to the following stages of the resulting methodology:

- First, we elicit, from the experts, the corresponding membership functions.
- Then, we need to process this additional information to come up with an appropriate control – just like automatic controllers of that time used crisp information to come up with a control.
- Finally, we need to incorporate this new information into the existing automatic controllers, to make them better.

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This was not easy sailing. The ideas were interesting, but immediately challenges appeared on all three stages – as a result of which, almost a decade went by before the first successful application appeared. Since the mid-1970s, fuzzy techniques have been successfully applied; see, e.g., [1], [7], [9], [10] – but some challenges remain. To better understand the remaining challenges – and to come up with ideas on how to overcome them – it is important to recall what were the previous challenges and how they were resolved.

II. Challenge Related to Eliciting Knowledge From Experts: What It Was and How It Was Resolved

Why is elicitation a challenge? At first glance, elicitation should be easy: we ask an expert for a number $m(x)$, he/she produces a number, what's the big deal?

But let us recall why we need these numbers in the first place. In the ideal world, an expert driver should be able to tell us exactly for how many millisecond and with what exactly force he brakes in a typical traffic situation – e.g., when on a freeway, a car 30 feet in front slowly down from 65 mph to 60 mph. In reality, a driver cannot provide this information. Instead of providing the exact duration of braking, the driver will just say "a short time" – and when you force the driver to provide a definite answer, he/she will provide different answers when asked the same question again.

Now you ask the same driver to describe his/her degree of confidence – e.g., that 100 msec is a short time – by an exact number. Clearly, the same driver who cannot provide the exact braking duration cannot provide the exact degree value either – and similarly, if you force him/her to provide a definite answer, he/she will provide different answers when asked the same question again.

If we use both (somewhat different) solicited values in the actual control, we get different results depending on which of these two values we use. In other words, in addition to eliciting knowledge, we also add, to our system, some undesired randomness. So, the challenge was – how to make elicitation process more natural? How to make sure that the elicitation captures the experts' knowledge – and does not add any unwanted randomness?

A natural solution to this challenge. As we have mentioned, eliciting numbers from people is not something Zadeh invented, it has been done, for many decades, in surveys. So why don't surveys have this problem?

Well, the answer is straightforward: surveys do not ask the user to pick up a real number from some interval. Instead, a survey usually provides a few selected numbers to choose from. For example, if the choice is – as with many US-based student evaluations – between 0, 1, 2, 3, and 4, then selecting 0 means, in effect, that the student's degree is somewhere between 0 and 0.5; selecting 1 means that it is somewhere between 1.5 and 2.5, etc. In other words, the surveys take into account that most people are unable to describe their opinion by an exact real number. Instead, they, in effect, allow the users to describe their degree not by a number, but by an *interval*.

So, a natural idea is to allow the user to describe his/her degree of confidence not by a single number $m(x)$, but rather by an *interval* $[\underline{m}(x), \overline{m}(x)]$. Such *interval-valued fuzzy techniques* was first proposed by Zadeh himself – and since then has led to many successful practical results; see, e.g., [7].

We can go further and ask: it is reasonable to expect the expert to produce exact values $\underline{m}(x)$ and $\overline{m}(x)$? Just like for braking duration, where the most natural expert's answer is a natural-language "fuzzy" term – that is described by a fuzzy set – similarly, the most natural way for the expert to produce a degree $m(x)$ is to use a natural-language fuzzy term – that is also described by a fuzzy set. This idea – where each $m(x)$ is itself a fuzzy set – is known as a *type-2 fuzzy set*. Such objects have also been successfully used in many application; see, e.g., [7] and references therein.

III. Challenge Related to Processing Fuzzy Information: What It Was and How It Was Resolved

So how do we process fuzzy information? Control does not mean that we simply use sensor readings, we first process them. What if instead of sensor readings, we have fuzzy expert opinions – how do we process them? Even if processing is very simple in the crisp case – e.g., if we add two numbers – how can we just add two fuzzy numbers?

Theoretical solution: Zadeh's extension principle. The main idea behind the answer to this question was also provided by Zadeh's himself – and it is now called *Zadeh's extension principle*. This solution grew from a simpler question related to the use of propositional connectives like "and", "or", and "not". Suppose that we have elicited, from the expert, a reasonable rule like "If the car in front is not very close and it slows down a little bit, then you brake a little bit". We can elicit, from the expert, a fuzzy set corresponding to "not very close" and a fuzzy set corresponding to "a little bit". However, what we need is to be able to describe, for each pair (d, f) of distance d and force f, to what extent the compound statement "the car in front is not very close and it slows down a little bit" is true. How can we do it?

Suppose that we consider N possible values of each quantity. So, to elicit two fuzzy sets, we need to ask $2N$ questions to the user. But to describe the degree corresponding to all pairs, we need N^2 – and if there are three conditions, N^3, etc. This clearly becomes not realistic. Since we cannot elicit the degrees of all such compound statements $A \,\&\, B$, a natural idea is to estimate such degrees based on available information, i.e., based on the degrees a and b of statements A and B. So, we need an algorithm $f_\&(a, b)$ that, given the degrees of confidence in statements A and B, estimates the degree of confidence in the statement $A \,\&\, B$. Such an algorithm is known as an *"and"-operation* or, for historical reasons, as a *t-norm*.

Similarly, we need an "or"-operation $f_\vee(a, b)$ – which is usually called a *t-conorm*. Under reasonable assumptions, the simplest t-norm is $\min(a, b)$ and the simplest t-conorm is $\max(a, b)$.

Now we are ready to explain Zadeh's extension principle. Let us recall the problem. We have an algorithm $f(x_1, \ldots, x_n)$. In the traditional data processing, we would apply this algorithm to n numbers – n values of the corresponding quantities. But what if instead of the exact values x_i, we only have expert opinions – described in terms of fuzzy sets $m_i(x_i)$? In this case, the result $y = f(x_1, \ldots, x_n)$ of applying the function $f(x_1, \ldots, x_n)$ to these fuzzy numbers should also be fuzzy. How can we find the corresponding degrees $m(y)$?

Each value $m(y)$ is the degree to which y is a possible value of the corresponding quantity. A value y is possible if there exist values $x_1, \ldots, x_n$ for which x_1 is a possible value of the 1st input, x_2 is a possible value of the 2nd input, etc., and $y = f(x_1, \ldots, x_n)$. The degree to which x_1 is possible is known – it is $m_1(x_1)$. Similarly, the degree to which x_2 is possible is $m_2(x_2)$, etc. So, if we use $\min$ to describe "and", the degree to which x_1 is possible *and* x_2 is possible, etc., is equal to $\min(m_1(x_1), \ldots, m_n(x_n))$. And what is "there exists"? It is nothing else but the infinite "or": it means that this property either holds for one of the tuples $(x_1, \ldots, x_n)$, or for another tuple, etc. So, if we use $\max$ for "or", we end up with Zadeh's formula:

$$m(y) =$$
$$\max(\min(m_1(x_1, \ldots, m_n(x_n)) : f(x_1, \ldots, x_n) = y). \quad (1)$$

Computational challenge. From the theoretical viewpoint, Zadeh's formula (1) is perfect: it is a natural formalization of the commonsense meaning, and it is, in principle, computable. The problem is that even for simplest algorithms like addition $f(x_1, x_2) = x_1 + x_2$, algorithm that, in the crisp cases, requires a single computational step, computing (1) means solving a complex optimization problem with a non-smooth (because of $\min$) objective function. Such problems usually require thousands (and more) computational steps. In other words, it means that the use of this formula slows down computations by several orders of magnitude.

What can we do?

A natural solution to this challenge. To solve this challenge, a natural idea is to take into account that in many cases, even when the information is fuzzy, we need to make a decision – and decisions are the ultimate objective of all practical situations: shall we recommend a surgery to a patient? shall we brake in a traffic situation? shall we buy or sell certain stock? shall we take a job offer? In general, if our degree of confidence is high, we should make this decision, if it is low, we may need to collect more information. Thus, a natural idea is to select some threshold value $\alpha \in (0, 1]$, and to make a decision if our degree of confidence is larger than or equal to α.

For each fuzzy set $m(x)$ and for each α, it makes sense to consider the set of all the values x for which the decision is made, i.e., the set $m_\alpha \stackrel{\text{def}}{=} \{x : m(x) \geq \alpha\}$. This set if known as the α-*cut* of the original fuzzy set. For most reasonable properties like "small", as we increase x from its smallest possible value, the degree $m(x)$ first grows, then it reaches its maximum (e.g., for $x = 0$), then starts decreasing. For such fuzzy sets, all α-cuts are intervals.

It is known that a fuzzy set is uniquely determined by its α-cuts: namely, $m(x) = \inf\{\alpha : x \in m_\alpha\}$. Because of the practical importance of α-cuts, it is desirable to describe all operations with fuzzy sets – including (1) – in terms of α-cuts. And, somewhat surprisingly, this idea – first described by Hung T. Nguyen in his 1978 paper [8] – solved the computational challenge.

Indeed, when is the degree $m(y)$ greater than or equal to α? The maximum of several numbers is larger than or equal to α if and only if at least one of them is $\geq \alpha$, i.e., when there exist values $x_1, \ldots, x_n$ for which $y = f(x_1, \ldots, x_n)$ and $\min(m_1(x_1), \ldots, m_n(x_n)) \geq \alpha$. And when is the minimum of several numbers larger than or equal to α? When each of these numbers is $\geq \alpha$, i.e., when $m_1(x_1) \geq \alpha$ and $\ldots$ and $m_n(x) \geq \alpha$. By definition of the α-cuts, this is equivalent to $x_1 \in m_{1,\alpha}$ and $\ldots$ and $x_1 \in m_{1,\alpha}$. Thus, the desired α-cut m_α is equal to

$$m_\alpha = \{f(x_1, \ldots, x_n) : x_1 \in m_{1,\alpha}, \ldots, x_1 \in m_{1,\alpha}\}. \quad (2)$$

The expression in the right-hand side of the equality (2) is known as the *range* of the function $f(x_1, \ldots, x_n)$ on the sets $m_{i,\alpha}$. This range is usually denoted by $f(m_{1,\alpha}, \ldots, m_{n,\alpha})$. So, Zadeh's formula (1) can be described as follows:

$$m_\alpha = f(m_{1,\alpha}, \ldots, m_{n,\alpha}).$$

In other words, processing fuzzy data can be reduced to processing intervals.

From the computational viewpoint, this is good news, because for processing intervals, there are many effective algorithms; see, e.g., [2], [4], [6] (which are, by the way, often under-used by the fuzzy community). For example, for addition $f(x_1, x_2)$, as one can easily check, adding two intervals $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$ simply means adding lower endpoint and adding upper endpoints:

$$[\underline{a}, \overline{a}] + [\underline{b}, \overline{b}] = [\underline{a} + \underline{b}, \overline{a} + \overline{b}].$$

So, when we need to add two fuzzy numbers, it is sufficient to add lower endpoints and upper endpoints of all the α-cuts. Yes, we need more operations than in the crisp case, but only because we have more values to process. The complexity of processing each pair of values is still the same – much faster than in solving the optimization problem (1).

And the same drastic decrease in computation time happens for all monotonic operations.

IV. Challenge Related to Combining Fuzzy and Other Techniques

What was (and is) this challenge: a brief reminder. The third challenge is to combine fuzzy techniques with more traditional methods. While there have been many successes in this direction, this is still largely work in progress.

In the past, such combination meant combining fuzzy control with traditional control methods, now it also means combining fuzzy techniques with deep learning, etc.

Maybe this is how we can solve this challenge. The fact that both two previously described major challenges were solved by using interval techniques makes us think that maybe interval techniques can also help us to combine fuzzy techniques with other methods.

Some preliminary results make us hopeful about interval techniques. For example, it turned out that the use of interval uncertainty helps to make deep learning more effective; see, e.g., [5]. This improvement is based on a very simple idea: that since training data comes with some accuracy, it makes no sense to perform computations with higher accuracy. For example, if we have measurement results with 1% accuracy, it only makes sense to compute intermediate results with 1% accuracy – and computing third and fourth etc. digits of these results is a waste of time.

There have also been efficient algorithms for propagating uncertainty – in particular, interval and fuzzy uncertainty – via deep learning [3].

V. What Next for Fuzzy?

Overall, we have all the reasons to be optimistic about the future of fuzzy. There are general reasons to be optimistic about the future of fuzzy. After all, the ultimate goal of all the inventions, of all the gadgets, of all the controls is to make us happy. And what makes us happy is not easy to describe in precise terms. It is not only objective characteristics like money: many high-income people are unhappy. This uncertainty of an objective is one of the main reasons why fuzzy techniques were so successful in the 1980s. For example, fuzzy rice cookers made the most tasty rice – and there is no easy way to describe it in precise terms. Fuzzy-controlled Sendai train had the smoothest ride – and there is no easy way to describe perceived smoothness in precise terms.

Researchers have been trying to formalize our perceptions for many decades, and it is clear that this is still – and will remain for a long time – a major challenge. As a result, techniques for translating human perceptions into precise computer-understandable form – and this is what generally understood fuzzy techniques are about – are needed and will be needed.

Will future fuzzy techniques be the same? Most probably not. Future techniques in the future fuzzy research will be definitely different from the technical ideas described in the pioneering 1965 paper by Zadeh – but this is OK: already a lot of what we consider fuzzy research goes way beyond these ideas, i.e., beyond degrees from the interval $[0,1]$ and simplest min and product "and"-operations. Just like now fuzzy techniques incorporate genetic and neural ideas, future fuzzy research will incorporate other ideas and techniques – but it will still be fuzzy, in the sense that it will help to transform imprecise ("fuzzy") human statements and perceptions into a precise computer-understandable terms.

Will we still focus on fuzzy conferences and journals? Probably not. Research-wise, we are very optimistic about fuzzy. A next question to ask is: Will fuzzy remain a separate discipline, with separate conferences and journals? Probably not.

For example, most engineers and scientists use calculus and mathematical analysis, but most applications of calculus are published outside conferences and journals on mathematical analysis. This already happened 60 years ago. You may recall that Zadeh's 1965 paper appeared in the journal titled *Journal of Mathematical Analysis and Its Applications*. In spite of the title – which was preserved for historical reasons – already in the 1960s, this journal was *not* a place where a typical researcher using calculus would submit his/her paper. Similarly, most papers using linear algebra are not published in specialized linear algebra journals – and the same is true for many other mathematical and computational techniques.

This is what Lotfi Zadeh always urged us to do: to consider fuzzy as one of the tools in a toolbox, to present more at conferences corresponding to the application areas. His vision led to Computational Intelligence Society and conferences that combine different tools. Let us continue to follow this vision.

The future is fuzzy. It is fuzzy as in "imprecise", but it is also fuzzy as in "fuzzy research will always be needed". So we strongly believe that the future is fuzzy. The future is fuzzy in the usual sense – that it is difficult to predict what will happen. And the future is also fuzzy in the optimistic sense – that research in fuzzy area, i.e., research aimed to describe imprecise ("fuzzy") human opinions and perceptions in precise computer-understandable terms will always be needed – and (hopefully) will always be successful.

References

[1] R. Belohlavek, J. W. Dauben, and G. J. Klir, *Fuzzy Logic and Mathematics: A Historical Perspective*, Oxford University Press, New York, 2017.

[2] L. Jaulin, M. Kiefer, O. Didrit, and E. Walter, *Applied Interval Analysis, with Examples in Parameter and State Estimation, Robust Control, and Robotics*, Springer, London, 2012.

[3] O. Kosheleva and V. Kreinovich, "How to propagate uncertainty via AI algorithms", *Proceedings of the 10th International Workshop on Reliable Engineering Computing REC'2024*, Beijing, China, October 26–27, 2024, pp. 5–12.

[4] B. J. Kubica, *Interval Methods for Solving Nonlinear Constraint Satisfaction, Optimization, and Similar Problems: from Inequalities Systems to Game Solutions*, Springer, Cham, Switzerland, 2019.

[5] C. Q. Lauter and A. Volkova, "A framework for semi-automatic precision and accuracy analysis for fast and rigorous deep learning", In: M. Cornea, W. Liu, and A. Tisserand (eds.), *Proceedings of the 27th IEEE Symposium on Computer Arithmetic ARITH 2020*, Portland, Oregon, June 7–10, 2020, IEEE Press, pp. 103–110.

[6] G. Mayer, *Interval Analysis and Automatic Result Verification*, de Gruyter, Berlin, 2017.

[7] J. M. Mendel, *Explainable Uncertain Rule-Based Fuzzy Systems*, Springer, Cham, Switzerland, 2024.

[8] H. T. Nguyen, "A note on the extension principle for fuzzy sets", *Journal of Mathematical Analysis and Its Applications*, 1978, Vol. 64, pp. 369–380.

[9] H. T. Nguyen, C. L. Walker, and E. A. Walker, *A First Course in Fuzzy Logic*, Chapman and Hall/CRC, Boca Raton, Florida, 2019.

[10] L. A. Zadeh, "Fuzzy sets", *Information and Control*, 1965, Vol. 8, pp. 338–353.

A personal view on the future of Fuzzy Sets within Computational Social Sciences

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***Abstract*— This paper emphasizes the added value that a fuzzy approach offers in problems where fuzzy uncertainty is unavoidable—whether due to inherently vague input information (e.g., linguistic data), the need for fuzzy reasoning for better human comprehension, or the preference for fuzzy outputs that allow decision-makers to consider context-specific factors and assume responsibility. Developing fuzzy information representation tools cannot be set aside by the overwhelming number of computational tools that do not capture the natural fuzziness human beings need to manage their own lives. In particular, we claim that fuzzy modeling is in the core of what we can call Computational Social Sciences.**

Keywords*— *fuzzy sets, explainability, computational sciences, social sciences, computational ethics, computational social sciences.

I. Introduction

The year 2017 marks a pivotal moment in the history of fuzzy modeling—not only due to the passing of Prof. Lotfi A. Zadeh but also because of the growing and unstoppable impact of computational technologies among fuzzy practitioners. A theory born (Zadeh, [1]) to solve daily life and deeply related to natural language and the so-called Artificial or Computational Intelligence, the fuzzy approach very fast became an alternative for addressing a wide range of problems. Under the impulse and the influence of the founder of Fuzzy Sets, Lotfi A. Zadeh, together with all the giant pioneers developing its theory and a number of key applications, great projects and even scientific institutions reached the support of research national foundations and prestigious industrial companies. The composition of the group of involved researchers has kept a quite constant equilibrium in time between Computer Scientists, Engineers and Mathematicians (a search within SCOPUS of the term "fuzzy" in the key words of scientific papers apparently shows around a 40% of papers coming from computer scientists, around 35% of papers coming from engineers and around a 25% of papers coming from mathematicians). But somehow it looks like that the impact of fuzzy-based technologies has lost part of the initial nerve.

In this paper we present a very personal view, based upon my own past research, to stress that of those small scientific pieces each one of us is polishing should should at the end allow the development of theoretical and computational tools that take advantage of the reality of fuzziness, and manage fuzziness. Of course there are still key problems we have not been able to solve. For example, the idea of fuzziness might create tension within experimental scientists, a context where it is demanded the possibility of repeating each experiment in order to check results. Such a demand can ne natural when results are crisp and at least we can obtain a probability distribution. But in many fields, and in particular Social Sciences, quite often this is not the case: input data might be fuzzy, information processing might suggest fuzzy argumentation and output might be preferably *fuzzy*.

Indeed, a relevant field where Fuzzy Set theory perfectly fits is Social Sciences, that come with specific problems, specific contexts and specific objectives that might suggest new tools and even a new field for research, Computational Social Sciences.

II. THE COMPETITION WITH PROBABILY

At the beginning of the history of Fuzzy Sets, strong criticisms came from the field of Probability (see, e.g., Zadeh [2]), basically claiming that there was no need of another uncertainty model, because Probability models were more than enough to deal with any kind of uncertainty. As pointed out in Montero [3], Kolmogorov's Probability model (see Kolmogorov [4]) strictly applies only to those events being isomorphic to a classical set (a crisp set) and some of the structures provided by classical Set Theory. Kolmogorov's model is this way a product of binary logic, where potential connectives can be naturally reduced to three operators: conjunction, disjunction and negation. A rather poor structure for fuzzy events (see, e.g., Montero [5,6]). Kolmogorov's probability model is the only appropriate model of uncertainty within a crisp, binary framework. But our world, the way human beings understand the world, is not crisp. Human society would be unbearable if our brain was ruled by Aristotelean binary logic.

Although there are many theories of Probability (see, e.g., Fine [7]), and Probability can be developed within more general frameworks (see, e.g., De Finetti [8]), it is important to realize that our natural language is essentially imprecise and subject to gradation (Zadeh [9]). Our natural language allows certain consistency tension. Arguments need to be explained and developed. This tension in our natural language is usually solved by some kind of precisiation (Zadeh [10]), using the human perception of the surrounding world applied to each specific context (Zadeh [11]). For example, some classical decision naking hard paradoxes can be softened and explained by means of a fuzzy approach (see, e.g., Montero [12]). In addition, Medicine has shown that emotions play a key role in human decision making (see Bechara *et al*. [13]) and that our brain is ready to represent information by means of gradable concepts (see Hsu *et al*. [14]).

I. Soft Reasonning, Explainabiltiy and Computational ethics

Of course we can always try an approach by means of Probability that might be good enough for certain purposes, but our brain naturally works with fuzzy uncertainty, and if we pursue an artificial intelligence tool, such a tool should be able to talk to human beings in the human language. The problem is that perhaps this well-known classical argument has been overwhelmed by some computational technologies that do not care so much about modeling, pursing prediction on the basis of sometimes massive recorded data, similarly to black boxes in decision making. Quite a number of fuzzy researchers have moved into this direction, and sometimes it looks like the final proof for any proposed approach is that its statistical behavior improves standard existing solutions, according to some *ad hoc* experiment. But such a statistical experimentation is usually a crisp experimentation, so it might be natural to expect that crisp quality measures will better fit for crisp outputs, that might be losing the natural gradation that human being perceives, as can be detected in our natural language. In some way, this is a consequence of an Artificial Intelligence focused on an *automatic* Decision Making instead of an *argumentative* Decision Support, where understanding the problem is part of the objective (taking a responsibility should require understanding).

Explainability should be therefore a must in any Artificial Intelligence tool, and this field is a natural field for fuzzy modeling. Any procedure assumed by a human being should be understood by such a human being in order to take the legal responsibility of the consequences of the final action, and therefore such an explanation should be in general given in terms of natural language. Of course a number of apparently intelligent tools can be developed to make automatic decisions in somehow controlled frameworks, where no true intelligence is really needed. But we should be aware that human beings learn from the past, but we do not properly learn from past crisp data, but from all the conceptual representations of that past in our brain, that are influenced by our past life and culture, including additional contextual information, that might take different formats (words, discourses, images, feelings, etc.)

But even an automatic (pseudo intelligent) decision maker will always need some small intelligence to detect that a particular circumstance might not fully fall within the designed expected standard. And this *meta analysis* might also require to establish some basic rules that limit action (e.g., some ethical principles). This is another natural field for fuzzy models, since *principles* should not be crisp in nature if we do not want to be blocked. Different principles will never by free of tension, and this tension between principles has to be wisely (case by case) managed, with prudence. Even Asimov's famous "Three Laws of Robotics", first introduced in *I, Robot* (1950), are not entirely crisp; their interpretation requires contextual reasoning, and the addition of a "Zeroth Law" (to avoid *harming humanity*) further underscores the inherent fuzziness of ethical decision-making. His "Three Laws of Robotics" might in principle appear as crisp consistent principles, but they eventually result in contradictions, and a careful thought will show they are not so crisp. I should acknowledge how complex I see to state ethical principles to limit potential objectives of any artificial intelligence, but I am even afraid that limiting the fields of application of artificial intelligence might not be a realistic option. In addition, the impact of a world partially ruled by artificial intelligences might affect human rights, a possibility not predicted in the 1948 Human Rights Declaration. Nevertheless, although we are quite late, something has to be urgently done, and reasoning with fuzzy concepts requires fuzzy logic.

Beside the above two fields, *computational explainabiltiy* and *computational ethics*, which imply the construction of (fuzzy) consistent discourses as decision making aid methodologies (to be developed within any knowledge management field), we should acknowledge that many hot issues in current Science are based upon the analysis of non-numerical data (texts, audios and videos). So, uncertainty models should move into this direction, where fuzziness is exceptionally positioned to treat as fuzzy what is or should be fuzzy. The main objective of a true Artificial Intelligence is to understand reality in order to help human being, and any human explanation is always context dependent. Our intelligent view of our past comes with a lot of knowledge about how the things work out. This was a main observation L.A. Zadeh use to refer when he explained how the idea of a fuzzy set came into his mind (see, e.g., Perry [15]). Zadeh initially guessed that Fuzzy Sets would exploit within social sciences, and I fully agree (see Montero [16]): the definitive success of fuzzy models will most probably come associated to the analysis of texts (see, e.g., Montero *et al*. [17]) and the analysis of current massive recording of all kind of data. For example, for the analysis of social network platforms. But always stressing knowledge management rather than modeling human acts (see Montero *et al*. [18]). Predicting without understanding is risky. We learned that in classical (experimental and therefore crisp) Science, the best theory is the one that gives best predictions. But when such predictions are not crisp, explainability becomes a key argument to assure understanding. Essentially, Fuzzy Sets is a knowledge representation tool and its natural framework in knowledge management.

IV. Final Comments

Since 1965 the fuzzy model has evolved into several directions (see, e.g., Bustince *et al*. [19]), and will keep evolving in the future to allow knowledge management (see also Montero *et al*. [20]). This natural evolution implies an explanability close to human reasoning, which is mainly based upon the natural language. Of course we have to take advantage of each new computational tool, that might suggest a revision of classical definitions (see, e.g., Montero *et al*. [21]). But human language and human brain interact under a learning processes that is never closed, continuously searching for appropriate approaches, tools and solutions, depending on information, objectives and capabilities. Our language and our brain are far from being rigid crisply defined classification tools (see, e.g., Amo *et al*. [22]).

ill, our language is consistent and very efficient despite its sential uncertainty, and human evolution proves that is has en the best solution to manage and communicate among ıman beings. At least meanwhile there was no Artificial telligence available tool, whose intensive presence might ange the structure of our brain of some future generations.

As a former president of both IFSA and EUSFLAT, I nclude with a call to the key scientific communities in fuzzy odeling—namely IFSA (including all its institutional embers) and the IEEE Computational Intelligence Society— continue fostering collaboration between theoretical and plied researchers. It is imperative that we reaffirm the nportance of fuzzy models in knowledge management, rticularly within the emerging field of Computational Social ciences.

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REFERENCES

L.A. Zadeh, "Fuzzy sets." Information and Control 8:338-353, 1965. DOI: https://doi.org/10.1016/S0019-9958(65)90241-X

L.A. Zadeh, "Is there a need for fuzzy logic?" Information Sciences 178:2751–2779, 2008. DOI: https://doi.org/10.1016/j.ins.2008.02.012

J. Montero and M. Mendel, "Crisp Acts, Fuzzy Decisions." In: S. Barro, A. Sobrino and A. Bugarin (eds), Advances in Fuzzy Logic, Universidad de Santiago de Compostela, Santiago de Compostela, Spain, 1998; pp. 219-238.

A.N. Kolmogorov, Foundations of the theory of probability. New York, Chelsea Publishing Company, 1950 (English translation based upon the original 1933 German monograph appeared in Ergebnisse Der Mathematik and a Russian translation published in 1936).

J. Montero, "Comprehensive fuzziness," Fuzzy Sets and Systems 20:79-86, 1986. DOI: https://doi.org/10.1016/S0165-0114(86)80033-1

J. Montero, "Extensive fuzziness." Fuzzy Sets and Systems 21:201-209, 1987. DOI: https://doi.org/10.1016/0165-0114(87)90164-3

T.L. Fine, Theories of Probability. Academic Press, New York, 1973.

B. de Finetti, Theory of Probability (2 vols.). Wiley, Chichester, 1974 and 1975.

[9] L. A. Zadeh, "Fuzzy logic = computing with words." IEEE Transactions on Fuzzy Systems 4:103-111, 1996. DOI: https://doi.org/10.1109/91.493904

[10] L.A. Zadeh, "Precisiated natural language (PNL)." AI magazine 25:74-92, 2004. DOI: https://doi.org/10.1609/aimag.v25i3.1778

[11] L.A. Zadeh, "A new direction in AI, toward a computational theory of perceptions". AI Magazine 22:73-84, 2001. DOI: https://doi.org/10.1609/aimag.v22i1.1545

[12] J. Montero, "The impact of fuzziness in social choice paradoxes." Soft Computing 12:177-182, 2008. DOI: https://doi.org/10.1007/s00500-007-0188-5

[13] A. Bechara, H. Damasio and A.R. Damasio, "Role of the amygdala in decision-making." Annals of the New York Academy of Sciences 985:356–369, 2003. DOI: https://doi.org/10.1111/j.1749-6632.2003.tb07094.x

[14] M. Hsu, M. Bahtt, R. Adolfs, D. Tranel and C.F. Camarer, "Neural systems responding to degrees of uncertainty in human decision-making." Science 310:1680-1683, 2004. DOI: https://doi.org/10.1126/science.1115327

[15] T. S. Perry, "Lotfi A. Zadeh [fuzzy logic inventor biography]." IEEE Spectrum 32:32-35, 1995. https://doi.org/10.1109/6.387136

[16] J. Montero, "Fuzzy logic and Science." Studies in Fuzziness and Soft Computing 243:67-77, 2009. DOI: https://doi.org/10.1007/978-3-540-93802-6_3

[17] J. Montero, G. Rodríguez-Caderot and P. Romero, "Logic, mathematics and consistency in literature: searching for Don Quixote's place." Nonlinear Systems and Complexity 18, 221-245, 2017. DOI: https://doi.org/10.1007/978-3-319-46164-9_11

[18] J. Montero, V. López and D. Gómez, "The role of fuzziness in decision making." Studies in Fuzziness and Soft Computing, 215:337-349, 2007. DOI: https://doi.org/10.1007/978-3-540-71258-9_16

[19] H. Bustince, E. Barrenechea, M. Pagola, J. Fernández, Z. Xu, B. Bedregal, J. Montero, H. Hagras, F. Herrera and B. De Baets, "A historical account of types of fuzzy sets and their relationships." IEEE Transactions on Fuzzy Systems 24:179-194, 2016. DOI: http://doi.org/10.1109/TFUZZ.2015.2451692

[20] J. Montero, H. Bustince, C. Franco, J.T. Rodríguez, D. Gómez, M. Pagola, J. Fernández and E. Barrenechea, "Paired structures and bipolar knowledge representation." Knowledge-Based Systems 100:50-58, 2016. DOI: https://doi.org/10.1016/j.knosys.2016.02.003

[21] J. Montero, R. González del Campo, L. Garmendia, D. Gómez, J.T. Rodríguez, "Computable aggregations." Information Sciences: 460:439-449, 2018. DOI: https://doi.org/10.1016/j.ins.2017.10.012

[22] A. Amo, J. Montero, G. Biging and V. Cutello, "Fuzzy classification systems." European Journal of Operational Research 156:495-507, 2004. DOI: https://doi.org/10.1016/S0377-2217(03)00002-X

60 Years of Fuzzy Sets and Systems and 30 Years of Genetic and Evolutionary Fuzzy Systems: An Exciting Personal Journey

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***Abstract*—Since their introduction by Lotfi A. Zadeh in the mid-1960s, fuzzy systems have proven to be a powerful paradigm for modeling and reasoning under uncertainty. Initially, the construction of fuzzy systems heavily relied on expert knowledge, with fuzzy rules and membership functions manually defined by human specialists. This approach emphasized linguistic modeling of complex systems. However, in the 1990s, the field witnessed a paradigm shift with the integration of machine learning techniques, enabling fuzzy systems to be automatically tuned or entirely learned from data. Methods such as fuzzy clustering, least squares estimation, neural networks, and evolutionary algorithms were increasingly employed to enhance the accuracy, adaptability, and scalability of fuzzy models. Among these, genetic and evolutionary fuzzy systems have emerged as a particularly flexible and robust framework, combining the interpretability of fuzzy logic with the optimization power of evolutionary computation. This approach lies at the core of the present contribution, revisiting its early developments and it current state from my personal viewpoint.**

***Index Terms*—fuzzy sets and systems, evolutionary algorithms, genetic and evolutionary fuzzy systems.**

I. Introduction

Fuzzy set theory, introduced by Lotfi A. Zadeh in 1965 [1], extended classical set theory by allowing gradual membership, offering a framework to handle imprecision and uncertainty. This laid the foundation for fuzzy systems (FSs), which model and control complex real-world processes that lack precise mathematical descriptions [2]. A landmark application was Mamdani's fuzzy logic controller (FLC) in the 1970s [3], which demonstrated how expert knowledge could be encoded through intuitive linguistic rules. Since then, FSs have become central to fields like control engineering, decision-making, system modeling, and pattern recognition.

In their early development, FSs were predominantly knowledge-based, with fuzzy rules and membership functions manually crafted by human experts [4]. This approach leveraged domain expertise to model system behavior in a transparent and linguistically interpretable way. However, as system complexity increased and expert knowledge became harder to elicit or scale, the limitations of purely manual design became evident. A significant milestone was the introduction of the Takagi-Sugeno-Kang (TSK) fuzzy systems in the mid-1980s [5], which enabled more precise modeling by allowing rule consequents to be expressed as mathematical functions, typically linear. This development bridged fuzzy inference with data-driven modeling and paved the way for a new generation of FSs that incorporate learning from data. Techniques such as (fuzzy) clustering [6], neural networks (leading to neuro-fuzzy systems) [7], and evolutionary algorithms (EAs) emerged to automatically identify, optimize, or adapt the structure and parameters of fuzzy models. In particular, genetic and evolutionary fuzzy systems (GEFSs) [8] apply evolutionary computation to optimize various aspects of FSs —including rule bases, membership functions, and even system architecture. GEFSs offer distinct advantages over other learning approaches, such as global search capabilities, flexibility in handling mixed or incomplete information, and the ability to maintain interpretability through tailored fitness functions. As a result, they have become a vital and active area of research within computational intelligence.

This contribution is devoted to revisit the creation and maturing of the GEFS research field from a personal perspective as the author played a role on its development. The contributions of GEFSs to computational intelligence will be reviewed and their current role in artificial intelligence explored.

II. The Early Days: 1991-95

The foundations of GEFSs were established in a very short period of time. The key point was to consider an evolutionary learning process to automate the design of the FS knowledge base (KB) from data. The first proposals involved learning: i) the data base (DB) by tuning a predefined definition [9]–[11]; ii) the linguistic rule base (RB) [12], [13]; and iii) the whole KB [14], [15].

By that time, two classical evolutionary learning approaches were considered to learn fuzzy rule bases: the *Michigan approach*, where a single rule was encoded in each chromosome and the whole population becomes the final RB definition as in [12], and the *Pittsburgh* approach where each chromosome encoded a whole RB definition as in [13]–[15].

The first monograph devoted to GEFSs was written in 1995 [16]. It concentrated exclusively on two specific topics: fuzzy classifier systems based on the Michigan approach and RB learning through genetic programming.

My first contact with the field was in March 1994. I had started working on fuzzy sets and systems during summer 1992, after the third year of my master's degree in computer science. Paco Herrera had been my professor in my second and third years. He proposed to a group of students to start a line of research in FLC inference system design. We successfully worked on the topic for two years, and the outcomes became my final degree project. We managed to publish three papers in different editions of the Spanish Fuzzy Conference (ESTYLF) —the first one in 1993 when I had just finished the fourth year of my degree—, two journal papers in BUSEFAL, and a pioneering paper in Fuzzy Sets and Systems [17]. During my degree project, I incorporated the genetic tuning method proposed by Herrera et al. [11] to our FLC designs. Then, by the beginning of 1995, we decided that my PhD dissertation, also advised by Paco Herrera, would be devoted to GEFSs and we started by writing the first survey on the topic in March 1995 reviewing the developments described in the current section [18].

III. The Establishment of the Discipline: 1996-2005

From those pioneering contributions, the interest of researchers on the topic significantly increased and a huge number of new proposals arose during the next decade. New evolutionary tuning proposals were proposed, considering different fuzzy membership function shapes and coding schemes, tuning of scaling functions, and especially incorporating interpretability constrains [19]. New research directions were created by the community including the proposal of new evolutionary learning approaches, evolutionary rule selection methods, GEFS designs for the interpretability-accuracy trade-off, multiobjective GEFSs, GEFSs to build fuzzy inference mechanisms, and GEFSs to learn extended fuzzy rule structures, among others.

The research group at the University of Granada made several influential developments in the field in this period. The first was the proposal of the *iterative rule learning* (IRL) approach, specifically tailored for FSs. Unlike Michigan- and Pittsburgh-style approaches, IRL decomposes the learning task into successive steps, evolving and adding a single fuzzy rule at a time. This strategy fosters interpretability, scalability, and ease of knowledge incorporation. Both MOGUL (Methodology to Obtain Genetic fuzzy rule-based systems Under the iterative rule Learning approach) [20] and SLAVE (Subtractive Learning Algorithm for Vague Environment) [21] were its most representative algorithms. These systems have since become seminal references in the GEFS literature and have influenced numerous follow-up works in classification, regression, and decision support. In fact, the proposal of MOGUL was the core of my PhD dissertation, defended in October 1997, becoming one of the first dissertations in the area.

The author later contributed with a novel evolutionary learning approach for GEFSs, the embedded KB learning. It was based on an evolutionary DB learning process which wraps a basic rule generation method. The EA derives the DB definition by learning components such as scaling functions/contexts, membership functions, and/or granularity parameters. A subsequent fuzzy rule generation method, which must be simple and efficient, derives the RB for the DB definition encoded in each chromosome, and some type of error measure is used to validate the whole KB obtained. An example of a GEFS based on this learning approach is [22].

The interpretability-accuracy trade-off played a significant role in the advancement of the discipline, as the flexibility of EAs were a superb capability to deal with the problem of designing both accurate and interpretable fuzzy models [23]–[25]. Multiobjective GEFSs [26] stood out as the most natural approach for this task as both requirements are clearly in conflict, becoming a very prolific topic in the GEFS research area. The multiobjective evolutionary learning/tuning process allows us to jointly consider the optimization of different accuracy and interpretability measures. The author also contributed to the area with one of the first proposals to learn the whole KB definition [27]. It jointly performed feature selection and fuzzy partition granularity learning within an embedded learning approach to obtain fuzzy classification systems with a good tradeoff between classification ability and RB complexity. He also introduced a tuning method for a real-world mobile robotics application [28].

A decade after the initial approaches emerged, the field's main achievements were comprehensively consolidated in the 2001 monograph by Cordón, Herrera, Hoffmann, and Magdalena [8]. This book is now a seminal reference in the field and is a required reading for anyone interested in pursuing research in this area. That is the reason why this book reports such a high number of citations (almost 1600 in Google Scholar) and has been adopted as a textbook in several parts of the world. In 2004, the author also contributed to the most cited survey in the area [29], with almost 1200 citations in Google Scholar, reviewing the developments described in this section. In the same year, he received an offer from Jerry Mendel, Chair of the IEEE Computational Intelligence Society's Fuzzy Systems Technical Committee, to create a Task Force on the topic. He founded the Genetic Fuzzy System (later GEFS) Task Force in 2004, chairing it until 2007. This resulted in the creation of the series of IEEE international symposia on GEFSs, for which six editions were held from the first one in Granada in 2005, two of them linked to the IEEE Symposium Series on Computational Intelligence (SSCI) multi-conference (GGEFS 2011 and GGEFS 2013).

IV. Killer Applications

Since their introduction in 1991, GEFSs have been successfully applied to a range of complex real-world problems where uncertainty, interpretability, and adaptability are crucial. Several developments stand out as:

1) A multiobjective genetic tuning approach for a bioaerosol detector optimizing accuracy (true/false positive rates) and interpretability (membership function similarity) [30]. The resulting fuzzy model infers air

safety in real time, addressing challenges like imbalanced data, time constraints, and false alarm reduction.

2) A genetic tuning method for a FLC in freight train speed control enhancing tracking accuracy and ride smoothness [31]. The approach enables offline customization for different track profiles while minimizing mechanical stress exerted on the couplers connecting the railcars.
3) A genetic Mamdani-type fuzzy classifier supporting early dyslexia diagnosis using imprecise, low-quality, graphical test data evaluated by experts [32]. Integrated into a web-based tool, it aids parents in identifying children who may require psychological assessment.
4) A tactical artificial intelligence (AI) framework developed by Thales Avionics and the U.S. Air Force for air combat combining fuzzy rule reasoning with genetic learning [33]. The system outperformed human pilots in simulations, showing GEFS's strength in real-time, interpretable decision-making for autonomous defense systems.
5) A many-objective type-2 GEFSs for field workforce optimization handling uncertainty while jointly optimizing multiple criteria [34]. Its impact includes £1M in productivity gains, 2500 tons of CO_2 saved, and potential prevention of over 100 serious injuries in UK roads.

In addition, the author has applied GEFSs to a wide range of real-world problems in various research projects and contracts. In the mid-1990s, he worked on the evolutionary identification of fuzzy models for electricity distribution networks in Asturias (Spanish Ministry of Research project with Hidroeléctrica del Cantábrico) [35]. He later developed multi-criteria FLCs for large-building HVAC systems using genetic algorithms (EU Joule project) [36]. By 2000, his work expanded into novel GEFS applications including Internet information retrieval [37], mobile robotics [28], and wind energy production forecasting (EDP Renewables contract).

Notably, he has led pioneering research on GEFSs for automating forensic identification using skeletal data, in collaboration with UGR's Physical Anthropology Lab and Prof. Caroline Wilkinson (Liverpool John Moores University, UK) [38]–[40]. Supported by multiple European and national projects —including the €1M MEPROCS project—, this work led to an international patent now exploited by the Panacea Cooperative Research company and commercialized in Mexico and South Africa. The system received the IFSA Award for Outstanding Applications of Fuzzy Technology in 2011.

V. And Now What?

More than three decades after their inception, GEFSs continue to play a vital role in the landscape of computational intelligence. Their enduring relevance stems from a unique combination of strengths: the ability to learn from data, maintain interpretability through linguistic rule-based models, and adapt flexibly through multi-objective optimization scenarios. As modern AI increasingly demands transparent, explainable, and human-centric solutions —particularly in safety-critical and ethically sensitive domains— GEFSs offer a compelling alternative to black-box models [41].

GEFSs own a distinctive set of capabilities that address some of the most pressing challenges in contemporary AI. Unlike deep learning models, which often operate as opaque black boxes, GEFSs produce rule-based models that are inherently understandable by humans. This makes them especially suitable for *high-stakes decision-making* in domains such as healthcare, finance, and autonomous systems, where transparency, trust, and accountability are critical. Furthermore, GEFSs are data-efficient —they can generate useful models even with limited or noisy data— thanks to the generalization capabilities of fuzzy logic and the robust search mechanisms of EAs.

Another important strength of GEFSs is their inherent flexibility and modularity. Multiobjective GEFSs can balance accuracy, complexity, and interpretability —something traditional machine learning methods struggle to address simultaneously. This is particularly important in real-world applications where simpler, more interpretable models are often preferred over marginal improvements in accuracy. GEFSs are also well-suited for hybridization, allowing integration with neural networks, ensemble learning, or probabilistic reasoning frameworks, making them a strong candidate for neuro-symbolic AI and explainable hybrid systems.

Moreover, the evolutionary component in GEFSs provides a natural mechanism for structural and parametric learning, enabling the system to automatically evolve fuzzy rules, membership functions, scaling functions, fuzzy inference systems, and system architectures. This capacity is crucial for tasks such as automated feature selection, system identification, and adaptive control in non-stationary or evolving environments —a key requirement in fields like robotics, smart manufacturing, and autonomous decision-making.

Hence, GEFSs represent a principled and versatile approach to AI that aligns well with emerging demands for explainability, adaptability, robustness, and human-centric design. Their foundational concepts remain not only valid but increasingly relevant as trustworthy AI continues to move beyond performance benchmarks toward real-world deployment, ethical accountability, and societal trust [42].

VI. Conclusions

Over the years, GEFSs became a very popular topic because of its high applicability in a wide variety of domains. Thousands of GEFS papers were published, special sessions organized in conferences, special issues edited in international journals, a specific series of bi-annual IEEE international symposia held, and several books written and edited (with a significant participation of the author in all those activities). Nowadays, several hundreds of GEFS researchers and practitioners can be found all over the world. His early start allowed the author to witness the historical development of GEFSs and to make key contributions to it, which positioned some of his publications as required readings for newcomers. This contribution has revisited the early foundations of the field and its later developments from the personal viewpoint of the author.

Far from being a legacy approach, GEFSs remain a foundational and forward-looking methodology, well-positioned to contribute to the next generation of intelligent systems. Moreover, their synergy with recent advances in machine learning, big data, and hybrid systems opens new avenues for innovation, from interpretable deep neuro-fuzzy architectures to automated knowledge extraction in dynamic environments.

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References

[1] L. A. Zadeh, Fuzzy sets, Information and Control 8 (3) (1965) 338–353.
[2] L. A. Zadeh, Outline of a new approach to the analysis of complex systems and decision processes, IEEE Transactions on Systems, Man, and Cybernetics 1 (1973) 28–44.
[3] E. H. Mamdani, Application of fuzzy algorithms for control of simple dynamic plant, Proceedings of the IEE 121 (12) (1974) 1585–1588.
[4] E. H. Mamdani, S. Assilian, An experiment in linguistic synthesis with a fuzzy logic controller, International Journal of Man-Machine Studies 7 (1) (1975) 1–13.
[5] T. Takagi, M. Sugeno, Fuzzy identification of systems and its applications to modeling and control, IEEE Transactions on Systems, Man, and Cybernetics 1 (1985) 116–132.
[6] S. L. Chiu, Fuzzy model identification based on cluster estimation, Journal of Intelligent & Fuzzy Systems 2 (3) (1994) 267–278.
[7] D. Nauck, F. Klawoon, R. Kruse, Foundations of Neuro-Fuzzy Systems, John Willey & Sons, 1997.
[8] O. Cordón, F. Herrera, F. Hoffmann, L. Magdalena, Genetic fuzzy systems: Evolutionary tuning and learning of fuzzy knowledge bases, Vol. 19, World Scientific, 2001.
[9] C. Karr, Genetic algorithms for fuzzy controllers, AI Expert 6 (2) (1991) 26–33.
[10] C. L. Karr, E. J. Gentry, Fuzzy control of ph using genetic algorithms, IEEE Transactions on Fuzzy Systems 1 (1) (1993) 46.
[11] F. Herrera, M. Lozano, J. L. Verdegay, Tuning fuzzy logic controllers by genetic algorithms, International Journal of Approximate Reasoning 12 (3-4) (1995) 299–315.
[12] M. Valenzuela-Rendón, The fuzzy classifier system: Motivations and first results, in: Parallel Problem Solving from Nature I, Springer, 1991, pp. 338–342.
[13] P. R. Thrift, Fuzzy logic synthesis with genetic algorithms, in: International Conference on Genetic Algorithms, 1991, pp. 509–513.
[14] D. Park, A. Kandel, G. Langholz, Genetic-based new fuzzy reasoning models with application to fuzzy control, IEEE Transactions on Systems, Man, and Cybernetics 24 (1) (1994) 39–47.
[15] A. Homaifar, E. McCormick, Simultaneous design of membership functions and rule sets for fuzzy controllers using genetic algorithms, IEEE Transactions on fuzzy systems 3 (2) (1995) 129–139.
[16] A. Geyer-Schulz, Fuzzy rule-based expert systems and genetic machine learning, Physica-Verlag, 1995.
[17] O. Cordón, F. Herrera, A. Peregrín, Applicability of the fuzzy operators in the design of fuzzy logic controllers, Fuzzy sets and systems 86 (1) (1997) 15–41.
[18] O. Cordón, F. Herrera, A general study on genetic fuzzy systems, in: Genetic algorithms in engineering and computer science, John Wiley and Sons Hoboken, NJ, USA, 1995, pp. 33–57.
[19] J. V. De Oliveira, Semantic constraints for membership function optimization, IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans 29 (1) (1999) 128–138.
[20] O. Cordón, F. Herrera, MOGUL: A methodology to obtain genetic fuzzy rule-based systems under the iterative rule learning approach, International Journal of Intelligent Systems 14 (11) (1999) 1123–1153.
[21] A. González, R. Pérez, Slave: A genetic learning system based on an iterative approach, IEEE Transactions on Fuzzy Systems 7 (2) (2002) 176–191.
[22] O. Cordón, F. Herrera, P. Villar, Generating the knowledge base of a fuzzy rule-based system by the genetic learning of the data base, IEEE Transactions on Fuzzy Systems 9 (4) (2001) 667–674.
[23] J. Casillas, O. Cordón, F. Herrera, L. Magdalena (Eds.), Interpretability Issues in Fuzzy Modeling, Springer, 2003.
[24] J. Casillas, O. Cordón, F. Herrera, L. Magdalena (Eds.), Accuracy Improvements in Linguistic Fuzzy Modeling, Springer, 2003.
[25] O. Cordón, A historical review of evolutionary learning methods for mamdani-type fuzzy rule-based systems: Designing interpretable genetic fuzzy systems, International Journal of Approximate Reasoning 52 (6) (2011) 894–913.
[26] M. Fazzolari, R. Alcala, Y. Nojima, H. Ishibuchi, F. Herrera, A review of the application of multiobjective evolutionary fuzzy systems: Current status and further directions, IEEE Transactions on Fuzzy systems 21 (1) (2012) 45–65.
[27] O. Cordón, F. Herrera, M. Del Jesus, P. Villar, A multiobjective genetic algorithm for feature selection and granularity learning in fuzzy-rule based classification systems, in: Joint IFSA-NAFIPS International Conference, Vol. 3, IEEE, 2001, pp. 1253–1258.
[28] R. Muñoz Salinas, E. Aguirre, O. Cordón, M. Garcia-Silvente, Automatic tuning of a fuzzy visual system using evolutionary algorithms: single-objective versus multiobjective approaches, IEEE Transactions on Fuzzy Systems 16 (2) (2008) 485–501.
[29] O. Cordón, F. Gomide, F. Herrera, F. Hoffmann, L. Magdalena, Ten years of genetic fuzzy systems: current framework and new trends, Fuzzy Sets and Systems 141 (1) (2004) 5–31.
[30] P. Pulkkinen, J. Hytönen, H. Koivisto, Developing a bioaerosol detector using hybrid genetic fuzzy systems, Engineering Applications of Artificial Intelligence 21 (8) (2008) 1330–1346.
[31] P. P. Bonissone, P. S. Khedkar, Y. Chen, Genetic algorithms for automated tuning of fuzzy controllers: a transportation application, in: Fuzz-IEEE, Vol. 1, IEEE, 1996, pp. 674–680.
[32] A. M. Palacios, L. Sánchez, I. Couso, Diagnosis of dyslexia with low quality data with genetic fuzzy systems, International Journal of Approximate Reasoning 51 (8) (2010) 993–1009.
[33] N. Ernest, D. Carroll, C. Schumacher, M. Clark, K. Cohen, G. Lee, Genetic fuzzy based artificial intelligence for unmanned combat aerial vehicle control in simulated air combat missions, Journal of Defense Management 6 (1) (2016) 144.
[34] A. Starkey, H. Hagras, S. Shakya, G. Owusu, iPatch: A many-objective type-2 fuzzy logic system for field workforce optimization, IEEE Transactions on Fuzzy Systems 27 (3) (2018) 502–514.
[35] O. Cordón, F. Herrera, L. Sánchez, Solving electrical distribution problems using hybrid evolutionary data analysis techniques, Applied Intelligence 10 (1999) 5–24.
[36] R. Alcalá, J. M. Benítez, J. Casillas, O. Cordón, R. Pérez, Fuzzy control of HVAC systems optimized by genetic algorithms, Applied Intelligence 18 (2003) 155–177.
[37] O. Cordón, F. de Moya, C. Zarco, A GA-P algorithm to automatically formulate extended boolean queries for a fuzzy information retrieval system, Mathware & Soft Computing 7 (2-3) (2000).
[38] O. Ibáñez, O. Cordón, S. Damas, J. Santamaria, Modeling the skull–face overlay uncertainty using fuzzy sets, IEEE Transactions on Fuzzy Systems 19 (5) (2011) 946–959.
[39] B. R. Campomanes-Alvarez, O. Ibáñez, F. Navarro, I. Alemán, M. Botella, S. Damas, O. Cordón, Computer vision and soft computing for automatic skull–face overlay in craniofacial superimposition, Forensic Science International 245 (2014) 77–86.
[40] B. R. Campomanes-Álvarez, O. Ibánez, C. Campomanes-Álvarez, S. Damas, O. Cordón, Modeling facial soft tissue thickness for automatic skull-face overlay, IEEE Transactions on Information Forensics and Security 10 (10) (2015) 2057–2070.
[41] A. Fernandez, F. Herrera, O. Cordon, M. J. del Jesus, F. Marcelloni, Evolutionary fuzzy systems for explainable artificial intelligence: Why, when, what for, and where to?, IEEE Computational Intelligence Magazine 14 (1) (2019) 69–81.
[42] N. Díaz-Rodríguez, J. Del Ser, M. Coeckelbergh, M. L. de Prado, E. Herrera-Viedma, F. Herrera, Connecting the dots in trustworthy artificial intelligence: From AI principles, ethics, and key requirements to responsible AI systems and regulation, Information Fusion 99 (2023) 101896.

Fuzzy Learning at 60: Future of Trustworthy AI in Healthcare and LLM

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***Abstract*—Fuzzy theory has long served as a cornerstone for modelling uncertainty and approximate reasoning in intelligent systems. As artificial intelligence advances toward interpretability and trustworthiness, the relevance of fuzzy systems is more significant than ever. In this contribution, we reflect on six decades of progress in fuzzy systems. We begin by revisiting foundational milestones, including Zadeh's seminal work on fuzzy sets and Mamdani's fuzzy controllers that revolutionized control systems. We then examine the neuro-fuzzy era, highlighting models like ANFIS and Takagi-Sugeno that blended data-driven learning with fuzzy inference. The last few decades have witnessed renewed momentum through integrations with deep learning, as well as advancements in type-2 to intuitionistic fuzzy frameworks. We place a particular focus on healthcare, where fuzzy systems support interpretable diagnosis, early detection of neurological disorders, and personalized decision support. These applications showcase fuzzy logic's strength in managing ambiguity in complex, high-stakes environments. Finally, we outline current limitations and offer recommendations for future research directions to further enhance the adaptability, scalability, and transparency of fuzzy systems.**

***Index Terms*—Fuzzy Logic, Neuro-Fuzzy Systems, Explainable Fuzzy Neural Network, Healthcare Decision Support, Fuzzy Machine and Deep Learning, Type-2 Fuzzy Systems, Interpretability, and Scalability.**

I. Introduction

Imagine being asked a simple question: Is the water hot? Although a thermometer might report a crisp numerical value, a human being is more likely to say "somewhat hot" or "a bit warm." This type of answer—imprecise, graded, and grounded in context—is how people naturally think and communicate. Traditional computing systems, built on binary logic, have long struggled to process this kind of soft, ambiguous information. Fuzzy set theory, introduced by Lotfi A. Zadeh in 1965 [1], was a bold response to this limitation—a mathematical framework that claimed not everything in the world can be cleanly divided into black and white. Fuzzy logic models the world as we experience it: full of gray areas, uncertainties, and degrees of truth. By assigning a membership value between 0 and 1 to the inclusion of an element in a set, fuzzy systems made it possible to reason about things like "tall people," "fast cars," or "high risk" in a structured way. But beyond modelling vagueness, fuzzy systems offer something even more powerful: alignment with human reasoning.

In 1969 Ruspini published a seminal paper on fuzzy clustering [2], integrating fuzzy sets with traditional unsupervised learning. Bezdek developed the general case of the fuzzy c-means model in 1973 [3], based on the well-known k-means clustering algorithm. Many branches of this tree grew from 1969 to 1993, including fuzzy possibilistic clustering [4].

The practical relevance of fuzzy logic was cemented in 1975 when Mamdani and Assilian introduced the first fuzzy logic controller, translating expert rules into a structured inference system [5]. These early successes showcased fuzzy logic's ability to operationalize human reasoning in control tasks without requiring precise mathematical models.

The ensuing decades saw fuzzy systems mature through integration with learning algorithms, leading to the *neuro-fuzzy era*. There were broadly two kinds of integration, *viz.* (i) Neuro-fuzzy systems (NFS) and (ii) Fuzzy neural networks (FNNs). While the NFS was a fuzzy system augmented by a neural network to enhance its flexibility, speed, and adaptability, the FNN was basically a neural network equipped with the capability of handling fuzzy information [6].

In recent years, fuzzy systems have reemerged at the forefront of Artificial Intelligence (AI) research through integration with deep learning and explainability. The convergence of fuzzy logic with deep neural networks—termed *fuzzy deep learning* (FDL)—has yielded models capable of handling noisy, imbalanced, and uncertain data while retaining interpretability [7]. Meanwhile, generalizations such as type-2 fuzzy sets [8], intuitionistic fuzzy sets [9], and hesitant fuzzy sets [10] have improved uncertainty modelling by capturing hesitation, ambiguity, and second-order vagueness.

The growing demand for responsible and interpretable AI

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has brought renewed attention to models that go beyond black-box accuracy and instead offer transparency, trust, and alignment with human reasoning [11]. In domains where decisions bear ethical or life-critical consequences—such as healthcare, law, and public policy—understanding how an AI system arrives at its conclusion is as important as the conclusion itself [12, 13]. Fuzzy logic, with its foundations in approximate reasoning and linguistic interpretability, provides a uniquely powerful framework for designing such human-aligned systems.

Notably, the emerging convergence of fuzzy reasoning with large-scale architectures such as large language models (LLMs) presents exciting new avenues for designing systems that are both capable and comprehensible [14]. Fuzzy logic offers potential tools to quantify uncertainty, enhance controllability, and infuse linguistic nuance in LLMs—helping bridge the gap between symbolic interpretability and sub-symbolic learning [15]. Motivated by these trends and opportunities, this commemorative note revisits foundational milestones, highlights transformative contributions, and outlines future challenges for fuzzy learning in a world increasingly shaped by responsible, scalable, and trustworthy AI.

II. Revolutionary Contributions of Fuzzy Theory

Over the past 60 years, fuzzy set theory has evolved from a framework for modeling vagueness into a widely applied paradigm for intelligent systems. Its development spans three key phases: foundational advances in fuzzy logic and control, the neuro-fuzzy era combining rule-based reasoning with learning, and recent integrations with deep learning and advanced uncertainty modeling. These innovations have expanded fuzzy systems' impact in critical domains such as healthcare and explainable AI. This section reflects on these interconnected phases of progress.

A. Early foundation

The inception of fuzzy set theory can be traced to Lotfi Zadeh's groundbreaking 1965 paper [1], which introduced the idea of sets with gradual membership. This conceptual leap allowed for a more natural representation of uncertainty and imprecision, especially in systems influenced by human language and perception. Soon after, fuzzy logic [16] was developed as a formal framework for approximate reasoning—capable of capturing the ambiguity inherent in real-world decision-making.

Building upon this foundation, Mamdani and Assilian [5] in 1975 pioneered fuzzy control systems that translated expert knowledge into interpretable IF-THEN rules. These Mamdani-type controllers were instrumental in demonstrating the practical viability of fuzzy logic. By the 1980s and 1990s, fuzzy control had become a dominant strategy in engineering applications, including automotive systems, consumer electronics, and public transportation, particularly in Japan [17]. These systems required no precise mathematical model of the plant, making them robust and adaptable in uncertain environments.

B. Neuro-fuzzy era

The 1990s and early 2000s saw the fusion of fuzzy logic with neural networks, leading to the development of neuro-fuzzy systems that combined the interpretability of fuzzy inference with the adaptability of machine learning. A landmark development in NFS for control systems was the Takagi–Sugeno–Kang (TSK) model [18], which introduced fuzzy rules with quantitative outputs, greatly improving the modelling of nonlinear processes. Soon after, Jang's Adaptive Neuro-Fuzzy Inference System (ANFIS) illustrated how neural network techniques could be used to automatically tune the parameters of a fuzzy inference system (FIS) [19]. These innovations blended the interpretability of fuzzy rules with the adaptive learning capabilities of neural networks, marking a shift from expert-defined fuzzy systems to data-driven *fuzzy modelling*.

The earliest research on modelling a fuzzy neuron, in perceptron, can be traced to Jim Keller in 1985 [20]. Studies in FNNs subsequently explored classification [21, 22] and rule generation [23, 24] paradigms. The well-known multilayer perceptron and Kohonen's self-organizing network were enhanced at the (i) input in terms of fuzzy linguistic functions, low, medium, and high, while the (ii) output modeled fuzzy membership between overlapping classes. The neuronal aggregation operator was replaced by a fuzzy aggregator for meaningful processing [25]. Researchers also used fuzzy objective functions for error minimization at output.

This era signalled a shift from purely rule-based to data-driven fuzzy modelling, making fuzzy systems more responsive to real-world data. The neuro-fuzzy paradigm opened up avenues in finance, speech recognition, pattern classification, and many other domains [6].

C. Fuzzy systems in the age of AI and LLMs

In the last few years, fuzzy systems have once again become central in the era of soft computing and hybrid intelligent systems. The integration of fuzzy logic with deep learning architectures—sometimes referred to as fuzzy deep learning—has enabled improved handling of uncertain, imbalanced, or noisy data [7]. The hybrid systems are increasingly relevant in mission-critical domains like medicine and autonomous systems. There have also been significant advancements in generalized fuzzy systems, including type-2 fuzzy sets, intuitionistic fuzzy sets, and hesitant fuzzy sets, which provide more nuanced modelling of uncertainty and hesitation [26]. These extensions allow for better capturing of ambiguity in both expert knowledge and data-driven scenarios.

Importantly, fuzzy systems have found a renewed purpose in explainable AI (XAI), where their transparent rule structures offer an interpretable alternative to black-box models. Applications in healthcare, smart environments, and decision support systems continue to benefit from the robustness and human-aligned reasoning capabilities of fuzzy logic [27].

Cutting-edge research has begun investigating the fuzzy logic behaviour exhibited by LLMs. For example, Singh [15] evaluated the extent to which LLMs are capable of performing fuzzy reasoning tasks. More recently, Chen et al.

[28] introduced a chaotic LLM-based educational question-answering system, wherein fuzzy control mechanisms dynamically fine-tune generation parameters using a Lee oscillator. This approach demonstrated enhanced performance in domain-specific educational applications.

These studies highlight an exciting frontier: integrating fuzzy reasoning directly within AI and LLMs to enhance uncertainty handling, and application-specific adaptation—paving the way for more trustworthy and human-aligned AI systems.

D. Applications in healthcare

Healthcare has emerged as one of the most critical and promising application domains for fuzzy systems. Trustworthy and interpretable systems are the need of the times. The inherent uncertainty, imprecision, and complexity of medical data make fuzzy logic an ideal tool for modelling, reasoning, and decision support in clinical settings. From diagnosis to prognosis and treatment planning, fuzzy systems offer a framework that accommodates ambiguity and expert knowledge in a transparent and interpretable manner [29].

One notable area of deployment has been in medical diagnosis and disease classification, where FIS are used to translate imprecise symptoms and lab values into actionable clinical insights. For instance, FNNs have gained traction in neuroimaging-based disease detection, particularly in complex neurological disorders such as Alzheimer's disease, where early detection is hindered by heterogeneity in biomarkers and symptom progression. Hybrid models combining fuzzy logic with deep learning have been proposed for EEG and MRI data analysis, offering both high accuracy and improved interpretability [30].

Fuzzy logic is also playing a role in personalized medicine, where patient-specific fuzzy rules help model heterogeneous treatment responses. Recent work on intuitionistic and type-2 fuzzy systems enables better modelling of patient hesitancy, treatment risk, and subjective assessment—factors crucial for building trustworthy AI in healthcare. Additionally, in the context of clinical decision support systems (CDSS), fuzzy systems provide an interpretable backbone for supporting complex decisions involving conflicting criteria, uncertain diagnoses, and multiple treatment pathways. Their rule-based structure makes them amenable to regulatory acceptance and clinical validation, compared to opaque black-box AI models.

III. Limitations and Future Research Directions

Despite significant advances in fuzzy systems and their applications, several limitations remain for future investigation. We outline below a few key challenges and prospective research directions, both general and domain-specific, that may shape next generation of fuzzy system models.

A. Scalability and rule explosion

A persistent limitation of classical fuzzy systems lies in their limited scalability to high-dimensional and large-scale problems. As the number of input features increases, the number of fuzzy rules tends to grow exponentially, resulting in what is commonly known as the *curse of dimensionality*. This increases computational burden and compromises one of the key strengths of fuzzy systems—interpretability.

Future direction: Research should focus on adaptive rule generation, sparse rule representations, and data-driven pruning techniques. Incorporating dimensionality reduction and neuro-symbolic optimization strategies could enable fuzzy systems to operate effectively in large-scale and high-dimensional environments while preserving their transparency. One promising approach involves the integration of randomized neural networks with FIS, which has shown potential in improving scalability while retaining interpretability [29], [31].

B. Adaptability in healthcare applications

Many fuzzy systems in healthcare rely on static, expert-defined rule bases, limiting their adaptability to diverse patient populations and evolving clinical knowledge. This rigidity may restrict clinical utility and long-term performance in dynamic healthcare environment.

Future direction: The development of personalized, adaptive fuzzy models is critical. Future systems should integrate online learning, real-time rule updating, and uncertainty modelling to tailor decision support to individual patients while maintaining interpretability and clinical relevance [30].

C. Integration with LLMs and deep architectures

Fuzzy logic has not been extensively integrated into large-scale deep learning frameworks, especially in natural language processing (NLP), LLMs and vision language models (VLMs). The absence of a seamless integration mechanism hinders its potential to improve explainability and trustworthiness in black-box models [14].

Future direction: Embedding fuzzy reasoning components within attention mechanisms or using fuzzy logic to quantify uncertainty in token-level importance offers a promising path. Hybrid fuzzy-transformer models could improve transparency and control in advanced AI systems, particularly in safety-critical applications. However the computational complexity, of such integration, needs to be kept in mind.

IV. Conclusions

As fuzzy systems evolve at the intersection of interpretability, uncertainty modelling, and data-centric learning, their role in shaping the next generation of AI becomes increasingly critical. The integration of fuzzy logic with deep architectures, the emergence of generalized frameworks, and their application to high-stakes domains such as healthcare collectively illustrate a paradigm shift—from isolated rule-based systems to adaptive, explainable, and trustworthy intelligent models.

Yet, this progress also reveals challenges that must guide future research. The need for scalable inference under high-dimensionality, dynamic adaptation in non-stationary environments, and seamless compatibility with LLMs and VLMs underscores the imperative for innovation in both theory and applications. Especially in healthcare, where interpretability and human-aligned reasoning are non-negotiable, fuzzy systems offer a unique foundation to build AI that is not only accurate but also transparent, reliable, and clinically meaningful. Advancing this frontier demands synergistic efforts across fuzzy theory, machine learning, and domain-specific knowledge, aimed at developing systems that reason under uncertainty while earning human trust. As AI faces increasing demands for responsibility and robustness, fuzzy logic remains an indispensable framework for shaping intelligent systems that are both powerful and principled.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] E. H. Ruspini, "A new approach to clustering," *Inf. Control*, vol. 15, pp. 22–32, 1969.

[3] J. C. Bezdek, "Fuzzy mathematics in pattern classification," *Ph.D. thesis, Cornell University, Ithaca, NY*, 1973.

[4] R. Krishnapuram and J. M. Keller, "A possibilistic approach to clustering," *IEEE Trans. on Fuzzy Systems*, vol. 1, pp. 98–110, 1993.

[5] E. H. Mamdani and S. Assilian, "An experiment in linguistic synthesis with a fuzzy logic controller," *International Journal of Man-Machine Studies*, vol. 7, no. 1, pp. 1–13, 1975.

[6] S. Mitra and Y. Hayashi, "Neuro-fuzzy rule generation: survey in soft computing framework," *IEEE Trans. on Neural Networks*, vol. 11, no. 3, pp. 748–768, 2000.

[7] Y. Zheng *et al.*, "The fusion of deep learning and fuzzy systems: A state-of-the-art survey," *IEEE Trans. on Fuzzy Systems*, vol. 30, no. 8, pp. 2783–2799, 2021.

[8] J. M. Mendel and R. B. John, "Type-2 fuzzy sets made simple," *IEEE Trans. on Fuzzy Systems*, vol. 10, no. 2, pp. 117–127, 2002.

[9] K. T. Atanassov and K. T. Atanassov, *Intuitionistic fuzzy sets*. Springer, 1999.

[10] V. Torra, "Hesitant fuzzy sets," *International Journal of Intelligent Systems*, vol. 25, no. 6, pp. 529–539, 2010.

[11] Y. Zheng *et al.*, "A systematic survey of fuzzy deep learning for uncertain medical data," *Artificial Intelligence Review*, vol. 57, p. 230, 2024.

[12] T. J. Ross, *Fuzzy logic with engineering applications*. John Wiley & Sons, 2005.

[13] S. Mitra, "Fuzzy MLP based expert system for medical diagnosis," *Fuzzy Sets and Systems*, vol. 65, pp. 285–296, 1994.

[14] M. Liu *et al.*, "The fusion of fuzzy theories and natural language processing: A state-of-the-art survey," *Applied Soft Computing*, p. 111818, 2024.

[15] S. Singh, "Are large language models good at fuzzy reasoning?" in *Proc. of 7th Intl. Conf. on Computational Intelligence and Intelligent Systems*, 2024, pp. 1–6.

[16] L. A. Zadeh, "Outline of a new approach to the analysis of complex systems and decision processes," *IEEE Trans. on Systems, Man, and Cybernetics*, vol. SMC-3, no. 1, pp. 28–44, 1973.

[17] J.-I. Horiuchi and M. Kishimoto, "Application of fuzzy control to industrial bioprocesses in japan," *Fuzzy Sets and Systems*, vol. 128, no. 1, pp. 117–124, 2002.

[18] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications to modeling and control," *IEEE Trans. on Systems, Man, and Cybernetics*, no. 1, pp. 116–132, 1985.

[19] J.-S. Jang, "ANFIS: adaptive-network-based fuzzy inference system," *IEEE Trans. on Systems, Man, and Cybernetics*, vol. 23, no. 3, pp. 665–685, 1993.

[20] J. M. Keller and D. J. Hunt, "Incorporating fuzzy membership functions into the perceptron algorithm," *IEEE Trans. on Pattern Analysis and Machine Intelligence*, vol. 7, pp. 693–699, 1985.

[21] S. K. Pal and S. Mitra, "Multi-layer perceptron, fuzzy sets and classification," *IEEE Trans. on Neural Networks*, vol. 3, pp. 683–697, 1992.

[22] S. Mitra and S. K. Pal, "Self-organizing neural network as a fuzzy classifier," *IEEE Trans. on Systems, Man and Cybernetics*, vol. 24, no. 3, pp. 385–399, 1994.

[23] ——, "Fuzzy multi-layer perceptron, inferencing and rule generation," *IEEE Trans. on Neural Networks*, vol. 6, pp. 51–63, 1995.

[24] ——, "Fuzzy self organization, inferencing and rule generation," *IEEE Trans. on Systems, Man and Cybernetics, Part A: Systems and Humans*, vol. 26, pp. 608–620, 1996.

[25] ——, "Logical operation based fuzzy MLP for classification and rule generation," *Neural Networks*, vol. 7, pp. 353–373, 1994.

[26] J. Lu *et al.*, "Fuzzy machine learning: A comprehensive framework and systematic review," *IEEE Trans. on Fuzzy Systems*, vol. 32, no. 7, 2024.

[27] J. Cao *et al.*, "Fuzzy inference system with interpretable fuzzy rules: Advancing explainable artificial intelligence for disease diagnosis—a comprehensive review," *Information Sciences*, vol. 662, p. 120212, 2024.

[28] H. Chen *et al.*, "Enhancing educational q&a systems using a chaotic fuzzy logic-augmented large language model," *Frontiers in Artificial Intelligence*, vol. 7, p. 1404940, 2024.

[29] M. Sajid *et al.*, "Neuro-fuzzy random vector functional link neural network for classification and regression problems," *IEEE Trans. on Fuzzy Systems*, vol. 32, no. 5, pp. 2738–2749, 2024.

[30] M. Tanveer *et al.*, "Fuzzy deep learning for the diagnosis of alzheimer's disease: Approaches and challenges," *IEEE Trans. on Fuzzy Systems*, vol. 32, no. 10, pp. 5477–5492, 2024.

[31] M. Sajid *et al.*, "Ensemble deep random vector functional link neural network based on fuzzy inference system," *IEEE Trans. on Fuzzy Systems*, vol. 33, no. 1, pp. 479–490, 2025.

The role of Fuzzy Logic in the era of Large Language Models

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***Abstract*—Society is undergoing a profound transformation, particularly in the way information is understood and accessed. Traditional search-based information retrieval is being replaced by Artificial Intelligence (AI) systems that proactively deliver responses tailored to users' informational needs. This paradigm shift has significantly impacted the design of information systems, moving away from conventional approaches—whether based on formal mathematical models or heuristic methods—towards deep learning algorithms. Fuzzy logic, which has played a fundamental role over the past six decades by enabling more natural and human-like interaction through the management of uncertainty and imprecision, now appears to be increasingly supplanted by large language models. These models inherently handle uncertainty, offering a new paradigm for user interaction that challenges the traditional relevance of fuzzy systems. However, despite their effectiveness in many areas, LLMs still present challenges that raise concerns about their deployment in sensitive domains, such as the management of biomedical data. Additionally, they pose significant issues related to sustainability and energy efficiency. These limitations prompt a reconsideration of alternative approaches, in which fuzzy logic may once again play a crucial role due to its capacity for handling uncertainty in a more interpretable and resource-efficient manner.**

***Index Terms*—Fuzzy logic, Computing with words, Large language models, sustainability, uncertainty, anniversary**

I. Introduction

For over six decades, fuzzy set theory—originally proposed by Lotfi A. Zadeh in the 1960s—has played a pivotal role in bridging the gap between human reasoning and computational systems [1]. Unlike classical binary logic, fuzzy logic allows for reasoning under uncertainty and imprecision, making it particularly well-suited for applications that require a more human-like approach to decision-making.

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Fuzzy sets theory has been successfully applied across a wide range of disciplines and industries [2]–[4]. In industrial automation, for instance, fuzzy controllers have been used to regulate temperature, pressure, and speed in complex systems where traditional control methods fall short. In consumer electronics, fuzzy logic has enhanced the performance of washing machines, air conditioners, and cameras by enabling adaptive behavior based on vague or incomplete input data.

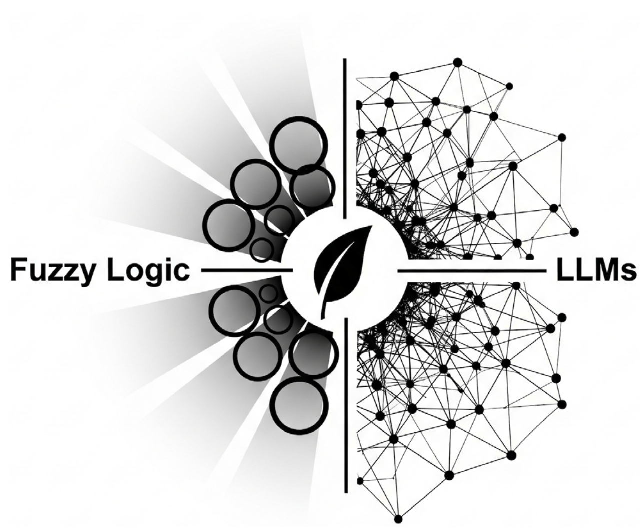

Fig. 1. Fuzzy Logic and LLMs (AI generated image).

In the domain of natural language processing (NLP), fuzzy logic has contributed to tasks such as sentiment analysis and semantic interpretation, where linguistic ambiguity is a central challenge. It has also been employed in meteorological forecasting, where it helps model the inherent uncertainty in weather predictions by incorporating imprecise variables like "partly cloudy" or "chance of rain."

In recent decades, the need to manage vast amounts of data and extract meaningful, actionable information has grown exponentially. The advent of the Internet of Things (IoT) and the proliferation of interconnected devices have led to an unprecedented surge in data generation. These developments have created a pressing demand for intelligent systems capable of processing, interpreting, and delivering relevant insights to end users in a comprehensible and context-aware manner [5].

One particularly illustrative and critical domain is bioinformatics—a field that inherently deals with massive volumes of complex, heterogeneous, and often uncertain data. From genomic sequencing to protein structure prediction and personalized medicine, bioinformatics applications require real-time data processing and interpretation. The results must then be communicated to researchers, clinicians, or patients in a way that is both accurate and tailored to their specific needs and levels of expertise [6].

In this context, fuzzy logic has proven to be a valuable tool. Its ability to model imprecise concepts and reason under uncertainty makes it especially suitable for biological systems, where data is often noisy, incomplete, or ambiguous. For example, fuzzy logic has been used to classify gene expression patterns, assess disease risk levels, and support decision-making in clinical diagnostics. Unlike rigid binary systems, fuzzy models can accommodate the nuanced and probabilistic nature of biological phenomena, offering more flexible and interpretable outputs [7].

More recently, in the last three years, the emergence of generative artificial intelligence has marked a significant turning point in the evolution of human–machine interaction. These systems, capable of producing original content from training data, have revolutionized the way machines communicate with humans. Among the most impactful developments in this domain are LLMs, which have demonstrated an unprecedented ability to generate coherent, context-aware, and human-like responses [8], [9].

A notable milestone in this trajectory is the performance of GPT-4.5, which reportedly achieved a 73% success rate on the Turing Test—a remarkable achievement that underscores the growing indistinguishability between machine-generated and human-generated language [10]. This level of fluency and contextual understanding has made communication with machines nearly seamless, opening new possibilities in education, customer service, healthcare, and beyond.

These technologies are increasingly perceived as natural successors to earlier computational paradigms. They offer faster, more scalable, and more user-friendly solutions for data processing and human interaction [11]. As a result, traditional systems—such as those based on fuzzy logic, and specially, the computing with words technique [12]—are facing a critical juncture. While these earlier approaches were designed to handle uncertainty and linguistic imprecision, they often require significant effort in terms of design, rule definition, and domain-specific tuning.

In contrast, LLMs can be deployed more rapidly and adapted to a wide range of tasks with minimal customization. This shift raises important questions for researchers and practitioners: Is it still worthwhile to invest in the development of interpretable, rule-based systems when generative models offer such compelling alternatives? Or should these technologies be seen as complementary, each suited to different types of problems and constraints?

The answer may lie in a hybrid approach, where the transparency and domain expertise embedded in fuzzy systems are combined with the generative power and adaptability of LLMs. Such integration could yield systems that are not only intelligent and efficient but also trustworthy and explainable [13].

II. Fuzzy Logic vs. LLMs

To address the current situation, it is appropriate to conduct a detailed analysis of the advantages and disadvantages of both approaches: the application of fuzzy set theory—integrated with paradigms such as computing with words, soft computing, and perception-based theory—to compute data and generate natural language text, and the use of large language models for data processing and summarization. Table I presents a comparative overview of their shared characteristics and distinctions.

TABLE I
Comparison between LLMs and Fuzzy Logic based systems

Aspect	**LLMs**	**Fuzzy Logic based systems**
Nature	Based on deep neural networks	Based on mathematical theory of fuzzy sets
Uncertainty Management	Implicit, through context and probabilistic modeling	Explicit, through degrees of membership
Interpretability	Low	High
Energy Consumption	High	Low
Typical Applications	Natural language processing, text generation, conversational agents	Fuzzy control systems, expert systems, automation
Adaptability	High, but prone to hallucinations	Limited, but more predictable

Advantages of using Fuzzy Logic:

- Handling of Uncertainty and Vagueness: Fuzzy logic excels in modeling linguistic variables such as “high temperature,” “moderate risk,” or “slightly better,” which are inherently vague and difficult to quantify using traditional binary logic.
- Interpretability: Fuzzy systems are rule-based and thus offer a high degree of interpretability.
- Adaptability: Fuzzy logic can be easily adapted to different domains by modifying the rule base.
- Low Computational Cost: Compared to deep learning models, fuzzy systems are lightweight and can be implemented on devices with limited computational resources, such as embedded systems or IoT devices.

However, despite these strengths, fuzzy logic also presents several limitations:

- Manual Rule Design: The creation of fuzzy rule sets often requires expert knowledge and can be time-consuming, especially in complex domains with many variables.
- Scalability Issues: As the number of input variables increases, the number of rules grows exponentially, leading to what is known as the "curse of dimensionality."

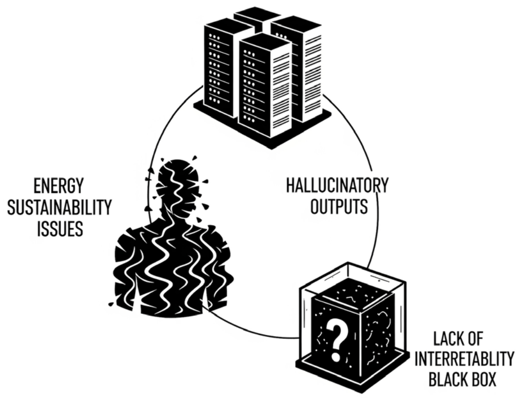

Fig. 2. LLMs limitations (AI generated image).

In contrast, LLMs advantages are:

- Natural Language Understanding and Generation: LLMs are capable of interpreting and generating text with a level of fluency and coherence that closely resembles human language.
- Versatility: Due to their training on extensive and diverse corpora, LLMs can adapt to a wide range of tasks without the need for task-specific retraining.
- Implicit Uncertainty Management: Although not explicitly designed for this purpose, LLMs handle ambiguity and vague contexts with surprising effectiveness.
- Accessibility and Scalability: Their integration into digital platforms has democratized access to advanced language processing tools, enabling widespread use across various domains.

Limitations of LLMs [14]:

- Lack of Interpretability: LLMs function as black-box models, making it difficult to understand how they arrive at specific conclusions—particularly problematic in critical fields such as medicine or law.
- Energy Consumption and Sustainability [15]: The training and deployment of LLMs require substantial computational resources, raising environmental and sustainability concerns.
- Ethical and Privacy Risks: LLMs may generate biased content, reinforce stereotypes, or mishandle sensitive data, posing ethical and privacy-related challenges.
- Dependence on Historical Data: Their knowledge is constrained by the data used during training, which can lead to outdated or contextually irrelevant outputs in novel or rapidly evolving situations.

III. A Case Study on the Integration of Fuzzy Set Theory and Large Language Models

In this study, we propose a hybrid system for monitoring blood glucose levels in users through an IoT-based sensor network [13]. The primary objective is to provide not only medical professionals but also family members and the users themselves with meaningful insights into their health habits. This is achieved by processing the collected data and generating *linguistic summaries* that highlight the most relevant events and trends in glucose measurements. Given the highly sensitive nature of biomedical data, our analysis indicates that full reliance on systems based solely on LLMs is currently not advisable because of this technology limitations.

To address this, we have developed a **hybrid architecture** that integrates expert knowledge modeled by healthcare professionals using **fuzzy set theory** and **Computing with Words** techniques. This expert-driven fuzzy model captures the nuanced understanding of blood glucose dynamics and feeds structured, interpretable knowledge into the AI system. The integration enhances the system's reliability and ensures that the outputs are both medically sound and user-friendly.

This approach achieves several key objectives:

1) **Expert-Guided Modeling:** By embedding domain expertise into the system, we ensure interpretability and trustworthiness in decision-making.
2) **Reduction of AI Hallucinations:** The structured fuzzy framework constrains the generative model, reducing the likelihood of producing inaccurate or misleading outputs.
3) **Improved Processing Efficiency:** The fuzzy layer filters and organizes data before it reaches the LLM, optimizing computational performance.
4) **Enhanced User Interaction:** The system delivers personalized, linguistically adapted summaries that improve user engagement and comprehension.

The experimental validation of this system was conducted in four distinct phases, as detailed in [13], demonstrating the feasibility and effectiveness of combining fuzzy logic with LLMs in real-world health monitoring scenarios:

- **Phase 1:**. *Data Acquisition and System Architecture:*
  i) *IoT Integration*: Glucose data is collected using the Freestyle Libre 3 sensor, which transmits data every 5 minutes via BLE/NFC to a smartphone app (xDrip+),
  ii) *Data Storage*: The data is stored in a MongoDB database and accessed via a RESTful API.
  iii) *Data Format*: Only essential fields (timestamp, UTC offset, and glucose value) are retained to ensure privacy.
- **Phase 2:**. *Expert-Guided Dataset Generation:*
  i) *Fuzzy Logic Modeling:* Medical experts define key glucose events (e.g., hypoglycemia, hyperglycemia, trends) using fuzzy logic to label and segment the time series.
  ii) *Protoform Generation*: Linguistic templates (protoforms) are created to describe glucose behavior using fuzzy quantifiers and membership functions.
  iii) *Quality Rules*: Summaries are designed to be concise,

non-redundant, and temporally structured.

iv) *Output*: Each TS is paired with a human-like summary, forming a dataset for training.

- **Phase 3.** . *Fine-Tuning of LLMs:*
  i) *Model Selection*: GPT-3.5, GPT-4o, and GPT-4o-mini were fine-tuned using OpenAI's API.
  ii) *Data Formatting*: Training data is structured in JSONL format with system, user, and assistant roles.

```
{
 "messages": [
  {"role": "system", "content":
     "[Domain], [Instruction],
     [Knowledge]"},
  {"role": "user", "content":
     "[TS data]"},
  {"role": "assistant", "content":
    "[Expected summary]"}
 ]
}
```

  iii) *Training Configuration*: Hyperparameters such as learning rate, batch size, and epochs were optimized to avoid overfitting.
- **Phase 4:**. *Prompt Engineering and Evaluation* (using the previously modeled knowledge with fuzzy logic):
  i) *Prompt Design*: Prompts are structured into four parts: a. *Header*: Introduces the task and TS data. b. *Domain*: Describes the medical context and relevant patterns. c. *Instruction:* Specifies the role (e.g., endocrinologist), output length, and time interval. d. *Knowledge*: Defines glucose ranges, time intervals, and quantifiers.
  ii) *Evaluation Framework (LLM-RawDMeth)*: A novel evaluation method was developed with 11 metrics across four dimensions: a. *Information Quality*: Accuracy, relevance, recall, correctness, reproducibility. b. *Thought Capabilities*: Understanding. c. *Communication Quality*: Clarity, accessibility, temporal dynamics. d. *Content Safety*: Hallucinations, assumption verification.

IV. Conclusions

As demonstrated in the presented use case, the integration of fuzzy set theory with LLMs offers a promising approach to mitigating the limitations inherent in each of these technologies. This synergy is particularly valuable in systems that require both natural language processing and robust uncertainty management, especially when dealing with sensitive or imprecise data.

By leveraging fuzzy logic to construct an expert knowledge model that informs the LLM, we enhance the system's reliability, reduce the risk of hallucinations, and improve the interpretability of the results. Conversely, the application of advanced data-handling techniques on the LLM side—such as fine-tuning and prompt engineering guided by the fuzzy knowledge model—enables improvements in processing speed, efficiency, and scalability.

Nevertheless, it is important to emphasize the issue of sustainability. Despite the design effort required, dedicated systems based on fuzzy logic remain significantly more energy-efficient than LLMs when solving many types of problems. This consideration is especially relevant in contexts where computational resources are limited or environmental impact is a concern.

In conclusion, fuzzy logic continues to play a vital role in technological solutions—not only in traditional control systems, where it has long been established, but also as a complementary component that enhances the performance and trustworthiness of LLM-based systems. Its contribution is particularly valuable in domains where interpretability, reliability, and human-centric reasoning are essential.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, 1965.

[2] R. R. Yager and L. A. Zadeh, *An Introduction to Fuzzy Logic Applications in Intelligent Systems*. USA: Kluwer Academic Publishers, 1992.

[3] L. A. Zadeh and R. A. Aliev, *Fuzzy Logic Theory and Applications*. World Scientific, 2018.

[4] J. Alcalá-Fdez and J. M. Alonso, "A survey of fuzzy systems software: Taxonomy, current research trends, and prospects," *IEEE Transactions on Fuzzy Systems*, vol. 24, no. 1, pp. 40–56, 2016.

[5] K. Rose, S. Eldridge, and L. Chapin, "The internet of things: An overview," *The internet society (ISOC)*, vol. 80, pp. 1–50, 2015.

[6] C. Huang, J. Wang, S. Wang, and Y. Zhang, "Internet of medical things: A systematic review," *Neurocomputing*, vol. 557, p. 126719, 2023.

[7] J. F. Gaitán-Guerrero, J. L. López, M. Espinilla, and C. Martínez-Cruz, "Towards an interpretable continuous glucose monitoring data modeling," *IEEE Internet of Things Journal*, p. 1–1, 2024.

[8] W. X. Zhao, K. Zhou, J. Li, T. Tang, X. Wang, Y. Hou, Y. Min, B. Zhang, J. Zhang, Z. Dong, Y. Du, C. Yang, Y. Chen, Z. Chen, J. Jiang, R. Ren, Y. Li, X. Tang, Z. Liu, P. Liu, J.-Y. Nie, and J.-R. Wen, "A Survey of Large Language Models," 2023.

[9] O. Nov, N. Singh, and D. Mann, "Putting ChatGPT's Medical Advice to the (Turing) Test: Survey Study," *JMIR Medical Education*, vol. 9, p. e46939, Jul. 2023.

[10] "Review: Large language models pass the turing test," vol. 2. [Online]. Available: https://s-rsa.com/index.php/agi/article/view/14697

[11] J. Liu, M. Yang, Y. Yu, H. Xu, T. Wang, K. Li, and X. Zhou, "Advancing bioinformatics with large language models: components, applications and perspectives," 2025.

[12] L. Zadeh, "Fuzzy logic = computing with words," *IEEE Transactions on Fuzzy Systems*, vol. 4, no. 2, pp. 103–111, 1996.

[13] J. F. Gaitán-Guerrero, C. Martínez-Cruz, M. Espinilla, D. Díaz-Jiménez, and J. L. López, "A novel fine-tuning and evaluation methodology for large language models on iot raw data summaries (llm-rawdmeth): A joint perspective in diabetes care," *Computer Methods and Programs in Biomedicine*, p. 108878, 2025.

[14] S. A. Khowaja, P. Khuwaja, K. Dev, W. Wang, and L. Nkenyereye, "Chatgpt needs spade (sustainability, privacy, digital divide, and ethics) evaluation: A review," *Cognitive Computation*, vol. 16, no. 5, pp. 2528–2550, 2024.

[15] Y. Ding and T. Shi, "Sustainable llm serving: Environmental implications, challenges, and opportunities : Invited paper," in *2024 IEEE 15th International Green and Sustainable Computing Conference (IGSC)*, 2024, pp. 37–38.

Holding Fuzzy Logic Steady in the New Era of General Artificial Intelligence

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***Abstract*—This note reports my opinion on the present of Fuzzy Logic within the context of current developments of Artificial Intelligence and Explainable AI. I recognize a critical moment for Fuzzy Logic, witnessed by a drop in published papers and, through a process of self-criticism, I highlight some issues in current research. Thinking at the future of research in the new era of General Artificial Intelligence, I propose some general guidelines to keep Fuzzy Logic on solid grounds.**

***Index Terms*—Fuzzy Logic, Artificial Intelligence, Explainable AI**

I. Backdrop

Since its original introduction by Lotfi Zadeh, Fuzzy Logic (FL), in its broad interpretation, fulfills a singular primary role: the representation and processing of imprecise and gradual information [1]. The primary utility of FL is to employ linguistic terms as symbolic representations of imprecise and gradual information, which are subsequently utilized, alongside their associated semantics, for computational purposes [2]. Therefore, FL is inherently suited to emulate human reasoning and perception-based information processing [3].

Throughout the 60-year development of FL, the history of Artificial Intelligence (AI) has undergone several paradigm shifts [4], which can be delineated as: (i) Expert Systems; (ii) Machine Learning; (iii) Deep Learning; (iv) General Intelligence. Undoubtedly, FL was significantly influenced by these paradigm shifts, particularly in the advancement of expert systems (such as fuzzy rule-based systems) and machine learning methodologies (including, but not limited to, fuzzy clustering, fuzzy rule induction, and fuzzy decision trees). A prominent attribute that substantiated FL methodologies was the *interpretability* of the resultant models, currently referred to as *ante-hoc* interpretability or model transparency [5].

The rapid advancements in Deep Learning (DL) present the AI community with a significant dilemma: while DL models demonstrate the capacity to address progressively sophisticated tasks with efficacy, their intrinsic complexity renders their internal mechanisms largely opaque and difficult to comprehend. This complex issue bears significant implications when AI systems are deployed within high-risk environments. Numerous countries, including those within the European Union, are formulating specific regulations mandating AI systems to furnish essential information that enables users to comprehend their decision-making processes and ensure human oversight [6]. Making AI explainable, initially intended as an academic desideratum [7], eventually gained momentum under the name of eXplainable AI (XAI) [8].

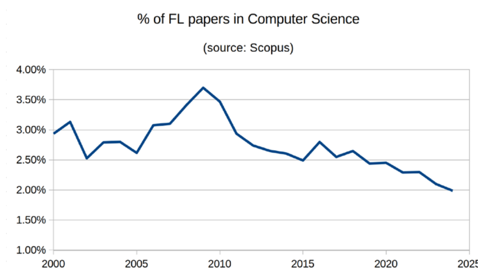

Fig. 1: Trend of published papers on FL in the subject area of Computer Science according to Scopus

In summary, on one side, FL is a key methodology for representing and processing human-centric information; on the other side, there is a strong demand for methods that explain AI decisions to humans. We could therefore expect FL to quickly converge into the XAI mainstream. **This has not happened yet**. For instance, the 2024 edition of the IJCAI conference[1], recognized as one of the premier conferences in the field of Artificial Intelligence globally, featured 13 works focused on XAI and none concerning FL. Similarly, the proceedings of ECAI 2024[2] comprise 22 papers on XAI, contrasted with merely two on FL. Additionally, within the proceedings of XAI-2024[3], an emerging conference series specifically dedicated to XAI, only 3 papers addressed FL out of a total of 97 published papers.[4] Overall, the proportion of published papers that include the term "fuzzy" in comparison to the total number of papers published within the subject area of Computer Science has been decreasing annually, a trend that has become particularly pronounced following the advent of DL around the year 2015, as illustrated in Fig. 1.

[1] https://www.ijcai.org/proceedings/2024/

[2] https://ebooks.iospress.nl/volume/ecai-2024-27th-european-conference-on-artificial-intelligence-1924-october-2024-santiago-de-compostela-spain-including-pais-2024

[3] https://xaiworldconference.com/2024/published-proceeding/

[4] In all the cases, a search of the words "fuzzy" and "explainab*" or "XAI" has been performed on the titles.

II. Self-criticism

Zadeh encountered numerous challenges in the dissemination of FL, which he persistently addressed and, to some extent, overcame through relentless advocacy and owing to demonstrable industrial applications evident in certain Japanese achievements.[5] The primary reason for these challenges appears to be cultural: Fuzzy Logic represented a revolutionary shift from the established understanding of what Logic ought to be, resulting in its challenging path to acceptance. My opinion is that this **cultural resistance is still present**.

Several underlying causes for this resistance are examined in a thought-provoking opinion paper by Hüllermeier, who presents compelling criticisms of the research in FL within the domain of Machine Learning (ML), which can be readily extended to the context of XAI [9]. Fundamentally, it is observed that there is an absence of rigorous practices, coupled with an aversion to contemporary AI methodologies. This situation causes publications on FL in ML to trail behind those presented in prominent academic venues. (See also my note at WILF 2019 [10].)

The domain of interpretability in FL modeling methodologies is not exempt from criticism. These criticisms are examined in a more systematic manner in an outstanding narrative review by Pickering *et al.*, who identified several potential reasons why the fuzzy modeling field remains absent from the mainstream discourse of interpretable ML, and by extension, XAI [11].

In both instances, the central message is that FL encounters challenges in extending its visibility beyond its own community. Nonetheless, FL should not be misconstrued as a religion: individuals should not be *converted* to adopt FL. Instead, the theoretical merits of FL should naturally surface, and its benefits in AI and XAI ought to be elucidated. To facilitate this occurrence, research in FL must be predicated upon indubitable foundations, with scholarly output adhering to the stringent standards requisite for dissemination in premier AI publications. Nonetheless, this necessitates a cultural transformation *within* the FL community, entailing a consensual and systematic body of theories, a standardized nomenclature, and a unified protocol for scientific advancement.

This self-criticism should not be interpreted as an objection to the FL community; rather, these are issues that naturally arise during the emergence of a new scientific field, and sixty years is not an extensive period for a discipline that challenges the fundamental principles of logic and uncertainty. However, as advancements in AI are progressing at an accelerated rate, the possibility that FL may not endure in the long term (except for some isolated groups) is significant. In my view, by directly confronting these challenges, the community can identify the appropriate pathway to maintain parity with other AI-related disciplines.

[5] https://time.com/archive/6703481/technology-time-for-some-fuzzy-thinking/

The field of XAI is relatively nascent, having gained prominence following the extensive adoption of DL methodologies. Due to its emergent status, XAI also contains inherent limitations. Freiesleben and König, in a recent paper, identified several critical issues regarding the current state of research on XAI [12]. Their work highlights various misconceptions within XAI that merit examination through the perspective of FL. Among these, a particularly pertinent issue for the FL community is the notion that XAI systems should give people explanations they find *intuitive*.

This misconception draws attention to the distinction between explanation and justification—a difference that is frequently disregarded. An explanation conveys the actual grounds for a decision, whereas justifications are just "good" reasons, i.e., reasons that make the decision *just*, thereby persuading individuals to place their trust in the model (for further discussion, refer to Freiesleben and König [12]). An illustrative example is the employment of a surrogate model that "explains" a black box by excluding sensitive features (e.g., gender or ethnicity) from the feature set. Consequently, we obtain a non-discriminatory explanation of the decision of a black box, which might have derived its output grounded in the omitted features.

FL models, which are frequently characterized by two distinct formal layers—namely, the symbolic layer encompassing linguistic terms, rules, etc., and the semantic layer comprising membership functions, operators, and other related elements—serve as potent instruments for elucidating model behavior in a manner that users can comprehend, assuming this is the intention (refer to other XAI misconceptions highlighted by Freiesleben and König). Nonetheless, the propensity to obfuscate intricate internal mechanisms with overly simplistic linguistic representations poses the risk of yielding outputs that merely justify rather than genuinely explain the model's results.

For instance, the term "fuzzy rules" typically refers to symbolic structures that fundamentally operate differently from classical implicative rules.[6] The primary distinction between these two types of rules lies in their truth values when the antecedent is false: fuzzy rules are evaluated as false, whereas classical rules are considered true. (There exist various interpretations of fuzzy rules; however, this discussion focuses on the Mamdani interpretation, which is predominantly utilized in the field of fuzzy logic literature.) Such differences necessitate alternative methods for the aggregation of rules and the process of inference.

We employ the term "fuzzy rule" due to its intuitive appeal, as it allows us to articulate the model within this formal structure (after all, who cares of a rule when its antecedent is false?). Nevertheless, in the broader context of the AI domain, the term "rule" carries a different connotation, being deeply rooted in Logic, which forms the foundation of conventional education. Consequently, any attempt to elucidate a FL model utilizing fuzzy rules often culminates in a mere justification,

[6] I am co-author of a book that uses this nomenclature.

thereby obscuring the inherent case-based reasoning mechanism.

In addition to fuzzy rules, other concepts intrinsic to FL may reveal the issue of justification as explanation. For instance, what is the interpretation of the "sharing degree" in fuzzy clustering? What is the significance of defuzzification, σ-count, strong fuzzy partition, rule weights, or some exotic t-norm/t-conorm pair? While these concepts indeed have definitions, a critical question persists as to whether their interpretations can be translated into a process of explanation that functions beyond mere justification.

III. Wishes for the future

A. Education

Students are frequently instructed in FL by initially introducing the definition of a fuzzy set, followed sequentially by fuzzy set operators, fuzzy rules, systems based on fuzzy rules, and subsequently fuzzy machine learning, among other topics. Numerous textbooks caution students regarding the limitless range of potential membership functions, t-norms, inference operators, and similar elements, as they *depend on the application*. It is my contention that this approach is inadequate for imparting the essential theory of FL, as it fails to address foundational principles adequately.

An educational framework for FL should commence with foundational concepts of graduality, corresponding to partial orders, and granularity, aligning with Possibility Theory. This framework should systematically develop the theoretical constructs and methodologies in a consistent manner. The semantics of fuzzy sets ought to be delineated through permissible operations, ensuring that the core knowledge remains straightforward and efficacious. Introduction of fuzzy models should be deferred, analogous to the later introduction of Bayesian Networks in the broader discipline of what may be termed "Probabilistic Logic".

Adopting a progressive methodology such as this will establish indisputable definitions of concepts and properties that remain beyond reproach, *by definition*, particularly outside the FL community. Students may discern that some of these concepts and properties are either analogous to those in non-FL domains or they genuinely present a novel perspective that can contribute significant value within AI and may, therefore, justifiably become integrated into the mainstream discourse of AI and XAI.

Numerous publications support this perspective, as evidenced by works such as Trillas and Eciolaza [13], Dubois and Prade [14], and Kruse *et al.* [15], among others. However, it is crucial to integrate these contributions into a coherent and comprehensive syllabus.

B. Software

Software serves as the principal catalyst for advancements in computational fields such as artificial intelligence. The advancement of AI in recent years has been significantly facilitated by software tools, including Weka, Python with its extensive library ecosystem, CUDA, and other specialized software. Within the domain of FL, notable efforts have been made to document software development contributions towards its progress, exemplified by the initiatives of the Task Force on Fuzzy Systems Software of IEEE CIS.[7]

The extensive collection of software compiled by the task force highlights the pressing need to underpin theoretical methodologies with practical implementations. Nonetheless, the majority of this software is tailored to specific fuzzy models, particularly fuzzy rule-based systems, whereas a more comprehensive framework is necessary to potentially construct such fuzzy rule-based systems.

The value of such a framework is multifaceted. Firstly, it facilitates the establishment of a common language, which can be utilized in educational contexts. It also empowers students to construct systems *de novo*, gaining comprehension and mastery over each component, evaluating alternatives, and pioneering new forms of systems. Secondly, a general software framework serves as a valuable asset in research by enabling reproducibility, interoperability, and the standardization of operations, thus supporting equitable comparison among models. By being constructed on foundational building blocks, the framework may enable the integration of theoretically robust concepts that are infrequently employed in modeling, such as relational equations, gradual numbers, etc.

C. Research protocols

To facilitate the integration of FL into mainstream research within the fields of XAI and AI, it is imperative to consider several crucial elements.

1) Motivation: A frequent issue in FL literature is the insufficient articulation of a compelling rationale for employing FL. However, a cultural bias persists within the AI community, wherein FL is often viewed with skepticism. It is therefore incumbent upon FL researchers to undertake additional efforts in demonstrating the unique contribution of FL to AI methodologies. This involves illustrating how FL facilitates features that are either unattainable or implemented in a less effective or efficient manner by alternative methodologies. Addressing this necessitates rigorous theoretical analysis of the proposed methods, alongside comprehensive experimental validation using both synthetic and *real* real-world datasets that extend beyond conventional public benchmarks.

2) Reproducibility: A significant challenge currently confronting AI is the phenomenon referred to as *reproducibility crisis* [16], which extends to all scientific disciplines in which AI is employed [17]. The primary catalyst for this crisis is the notably elevated adaptability of AI models, coupled with an overall deficiency of rigor in their conceptualization. Moreover, numerous AI models exhibit heightened sensitivity to precise coding, stochastic initialization, and hyperparameter configurations.

These issues may similarly impact FL models. As reproducibility serves as a cornerstone of scientific progress, it is imperative to endeavor to mitigate the risk of disseminating

[7] https://sci2s.ugr.es/TF-FSS

results that lack scientific validity. This objective can be achieved by implementing and mandating protocols that ensure the reproducibility of results and methods, in line with the standards adopted by other scholarly communities.[8]

Prior to the implementation of any policy, however, it is imperative that the ethos of reproducibility is ingrained within the community. This necessitates the establishment of a unified nomenclature and conceptual framework, coupled with rigorous mandates for the comprehensive documentation of methodologies and the institution of standardized validation protocols for experimental outcomes.

D. Beyond rules

Contemporary research in AI is progressing toward innovative forms of data representation, which are increasingly prevalent in information systems. At present, unstructured or semi-structured textual data, images (including hyper-spectral or three-dimensional), and graphs are ubiquitous. Furthermore, emerging or revitalized ML tasks, such as reinforcement learning, transfer learning, and generative modeling, are gaining prominence. In the domain of XAI, particularly, causal modeling has emerged as an essential methodology for furnishing robust explanations.

Frequently, complex data and tasks are managed by opaque models, which yield accurate yet potentially irreproducible results in high-stakes application contexts. In such scenarios, I align with Rudin's perspective, advocating against attempts to elucidate black-box models, and instead supporting the preference for interpretable models [18]. FL can significantly contribute to the development of interpretable models for critical applications, yet it is imperative to remain current with the advancements in related research domains regarding data and tasks. Nonetheless, this does not warrant indiscriminate *fuzzification*; motivation and reproducibility should consistently be our guiding principles.

IV. Take-home message

The advent of General Artificial Intelligence is exerting a considerable impact on scientific research and society as a whole. While this development presents substantial opportunities for advancements in scientific inquiry, it also poses existential threats, potentially leading to the replacement of science with pseudoscience. XAI transcends being merely a methodological approach; it embodies a commitment to maintaining oversight over AI technologies, ensuring that human-centric considerations remain at the forefront of decision-making processes. The future of FL lies in this domain; if venturing beyond established comfort zones is necessitated, preparedness is imperative.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, Jun. 1965. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S001999586590241X

[2] L. Zadeh, "Fuzzy logic = computing with words," *IEEE Transactions on Fuzzy Systems*, vol. 4, no. 2, pp. 103–111, May 1996. [Online]. Available: https://ieeexplore.ieee.org/abstract/document/493904

[3] L. A. Zadeh, "Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic," *Fuzzy Sets and Systems*, vol. 90, no. 2, pp. 111–127, Sep. 1997. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0165011497000778

[4] V. Dhar, "The Paradigm Shifts in Artificial Intelligence," *Communications of the ACM*, vol. 67, no. 11, pp. 50–59, Nov. 2024. [Online]. Available: https://dl.acm.org/doi/10.1145/3664804

[5] L. Magdalena, "Fuzzy Systems Interpretability: What, Why and How," in *Fuzzy Approaches for Soft Computing and Approximate Reasoning: Theories and Applications: Dedicated to Bernadette Bouchon-Meunier*, M.-J. Lesot and C. Marsala, Eds. Cham: Springer International Publishing, 2021, pp. 111–122. [Online]. Available: https://doi.org/10.1007/978-3-030-54341-9_10

[6] A. Bellogín, O. Grau, S. Larsson, G. Schimpf, B. Sengupta, and G. Solmaz, "The EU AI Act and the Wager on Trustworthy AI," *Communications of the ACM*, vol. 67, no. 12, pp. 58–65, Dec. 2024. [Online]. Available: https://dl.acm.org/doi/10.1145/3665322

[7] R. S. Michalski, "A theory and methodology of inductive learning," *Artificial Intelligence*, vol. 20, pp. 111–161, 1983. [Online]. Available: http://linkinghub.elsevier.com/retrieve/pii/0004370283900164

[8] A. Barredo Arrieta, N. Díaz-Rodríguez, J. Del Ser, A. Bennetot, S. Tabik, A. Barbado, S. Garcia, S. Gil-Lopez, D. Molina, R. Benjamins, R. Chatila, and F. Herrera, "Explainable Artificial Intelligence (XAI): Concepts, taxonomies, opportunities and challenges toward responsible AI," *Information Fusion*, vol. 58, pp. 82–115, Jun. 2020. [Online]. Available: https://linkinghub.elsevier.com/retrieve/pii/S1566253519308103

[9] E. Hüllermeier, "Does machine learning need fuzzy logic?" *Fuzzy Sets and Systems*, vol. 281, pp. 292–299, Dec. 2015. [Online]. Available: https://linkinghub.elsevier.com/retrieve/pii/S0165011415004133

[10] C. Mencar, "Looking at the Branches and Roots," in *Fuzzy Logic and Applications*, R. Fullér, S. Giove, and F. Masulli, Eds. Cham: Springer International Publishing, 2019, vol. 11291, pp. 249–252. [Online]. Available: http://link.springer.com/10.1007/978-3-030-12544-8_22

[11] L. Pickering, K. Cohen, and B. De Baets, "A Narrative Review on the Interpretability of Fuzzy Rule-Based Models from a Modern Interpretable Machine Learning Perspective," *International Journal of Fuzzy Systems*, Mar. 2025. [Online]. Available: https://doi.org/10.1007/s40815-025-02022-z

[12] T. Freiesleben and G. König, "Dear XAI Community, We Need to Talk!: Fundamental Misconceptions in Current XAI Research," in *Explainable Artificial Intelligence*, L. Longo, Ed. Cham: Springer Nature Switzerland, 2023, vol. 1901, pp. 48–65. [Online]. Available: https://link.springer.com/10.1007/978-3-031-44064-9_3

[13] E. Trillas and L. Eciolaza, *Fuzzy Logic: An Introductory Course for Engineering Students*, ser. Studies in Fuzziness and Soft Computing. Cham: Springer International Publishing, 2015, vol. 320. [Online]. Available: https://link.springer.com/10.1007/978-3-319-14203-6

[14] D. Dubois, H. Prade, D. Dubois, and H. Prade, Eds., *Fundamentals of Fuzzy Sets*, ser. The Handbooks of Fuzzy Sets Series. Boston, MA: Springer US, 2000, vol. 7. [Online]. Available: http://link.springer.com/10.1007/978-1-4615-4429-6

[15] R. Kruse, S. Mostaghim, C. Borgelt, C. Braune, and M. Steinbrecher, *Computational Intelligence: A Methodological Introduction*, ser. Texts in Computer Science. Cham: Springer International Publishing, 2022. [Online]. Available: https://link.springer.com/10.1007/978-3-030-42227-1

[16] M. Hutson, "Artificial intelligence faces reproducibility crisis," *Science*, vol. 359, no. 6377, pp. 725–726, Feb. 2018. [Online]. Available: https://www.science.org/doi/10.1126/science.359.6377.725

[17] P. Ball, "Is AI leading to a reproducibility crisis in science?" *Nature*, vol. 624, no. 7990, pp. 22–25, Dec. 2023. [Online]. Available: https://www.nature.com/articles/d41586-023-03817-6

[18] C. Rudin, "Stop explaining black box machine learning models for high stakes decisions and use interpretable models instead," *Nature Machine Intelligence*, vol. 1, no. 5, pp. 206–215, May 2019. [Online]. Available: https://www.nature.com/articles/s42256-019-0048-x

[8] See, e.g., ACM Artifact Review and Badging https://www.acm.org/publications/policies/artifact-review-and-badging-current.

Possibility theory - A tool for the future

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***Abstract*—A plea for possibility theory as a major spin-off from fuzzy set theory, emphasizing its representation power, and the variety of its applications.**

I. Introduction

For the 50th anniversary of fuzzy sets, we wrote a series of articles [1] covering the history (1-2), the legacy (3, 8) of the idea of fuzzy sets, as well as some aspects of possibility theory (4-5) and some relationships with other information processing settings (6-7).

As recalled in [1]-(3), membership grades may refer to an idea of distance (e.g., as in fuzzy clustering), or may be related to preference when expressing degrees of satisfaction, or yet may express degrees of plausibility when related to uncertainty. As pointed out very early by Zadeh [2] fuzzy sets may represent possibility distributions, which are the basic building block of possibility theory [3].

Until now possibility theory has mainly addressed the uncertainty semantics (interpreting possibility as plausibility and necessity as certainty), and to a lesser extent, the modeling of preferences (interpreting possibility as feasibility, and necessity as priority).

II. Possibility theory: Modeling epistemic uncertainty and preferences

The first elements of possibility theory, pioneered in the late 1940's by the theory of potential surprise of the English economist G. L. S. Shackle, were independently reinvented in the late 1970's by L. A. Zadeh who focused on the idea of graded possibility in relation with the modeling of linguistic information.

Possibility theory offers a particularly rich setting for the representation of incomplete information, thanks to the existence of four set functions that can be associated to a possibility distribution: a pair made of a weak possibility measure and its dual strong necessity measure, together with a pair made of a strong possibility function and its dual weak necessity function.

Possibility measure axioms use the maximum operation instead of the sum for probabilities. Thus, possibility theory may be fully qualitative or may have a quantitative flavor depending on the scale used.

Qualitative possibility theory is instrumental when developing possibilistic logic and non-monotonic reasoning [4]. Conditioning is then based on minimum rather than product [3]. It is also useful in decision under uncertainty where qualitative counterparts of the expected utility criterion have been proposed and axiomatized in the style of Savage framework of acts. These qualitative criteria based on $\min$ and $\max$ operations can be refined by using leximin and leximax orders [5]. Such refined criteria then satisfy the same properties as expected utility (which uses sum and product).

In the quantitative setting, possibility and necessity measures offer the simplest non trivial system of upper and lower probabilities and represent imprecise consonant information (while probability functions are tailored to precise and scattered data). Quantitative possibility theory is then useful for some aspects of statistical reasoning; for instance possibility distributions are useful to describe the dispersion of probability measures [6]. From another point of view, likelihood functions can be modeled by possibility distributions. See [7] for details. Numerical possibility theory has proved instrumental in signal processing [8], especially kernel-based approaches where it avoids the choice of a precise kernel. Viewed as modeling a convex set of probability functions, a possibility distribution may look poorly expressive as it models probability intervals of the form $[a, 1]$ or $[0, b]$ only. More expressive representations (still computationally simple) can be obtained using pairs of possibility distributions [9].

It is thus clear that possibility theory then stands halfway between logical and probabilistic representation frameworks. This fact is all the clearer as, besides possibilistic logic, there are graphical representations of possibility distributions akin to Bayesian networks [10], where conditioning is based on minimum or on product. Possibility theory is actually the proper setting for handling epistemic uncertainty, and dealing with incomplete information.

Finally, in the qualitative setting, it has been shown that general monotonic set-functions, known as capacities or fuzzy measures, can be interpreted as families of possibility measures [11], which enable strong similarities between qualitative capacities and imprecise probabilities as well as belief functions to be laid bare. These ideas have been applied to the modeling of pessimism and optimism in qualitative decision criteria (such as Sugeno integrals), as well as information fusion [12].

Possibility theory has seen a wide range of applications over the last thirty years [13], including representation of, and reasoning with, fuzzy rules, non-classical logics, decision under uncertainty, preference and desire modeling, risk analysis, databases, or machine learning. Other applications deal with data and expert knowledge fusion, constraint propagation, computations with imprecise or fuzzy quantities, scheduling, spatial interpolation (kriging), data reconciliation, and so on (see for [14] for references). The reader is referred to [15] for an extensive and recent review of possibility theory developments in reasoning, statistics and learning in an AI perspective. More recently the paper [4] offers an overview of 40 years of research in possibilistic logic and related logics.

III. Possibility Theory: A tool for a sober AI?

For many people there is an impassable gap between the AI that developed expert rules systems and the AI focused on machine learning based on neural networks. Such a claim is highly debatable since we can move from a min-max matrix calculus encoding uncertain rules in parallel and in cascade to a neural net [15], [16]. Such a starting point has led recently to an impressive, comprehensive, possibilistic neuro-symbolic approach which propagates possibility distributions (rather than probility distributions) [17].

This is part of a general concern to develop a more qualitative (and perhaps more physiologically plausible) view of neural nets [18]. Indeed McCulloch and Pitts' numerical formal neurons are perhaps worth questioning. Thus recently, Strauss, Rico *et al.* [19] have proposed a formal neuron made of the sum of four weighted maximum operations.

More qualitative models are unlikely to perform as well as sum-and-product neuro-probabilistic systems. However, if sufficiently developed, they might reach reasonably good performance acceptable in some daily life applications, with a lower computational cost. This is why studying $\max$-$\min$, $\max$-sum, or $\max$-$\min$-product systems is worth of interest (without even speaking of representational power).

Moreover, $\max$-pooling and $\min$-pooling are well-known layers in convolutional neural networks. No need to say that this kind of operations are similar to the computation of the weak possibility and the strong possibility function from a possibility distribution. Recent works [20] has shown their good behavior for handling embeddings as epistemic states.

IV. Conclusion

Possibility theory deserves attention as a representational and a computational tool to be further developed for uncertainty handling in AI and other applications. Until now possibility theory has only been developed by a small number of researchers. We hope that more scholars will use it in the future, due to its versatility and its low computational complexity.

References

[1] D. Dubois and H. Prade. "Articles written on the occasion of the 50th anniversary of fuzzy set theory," Research Report n°RR–2015–11–FR, IRIT: Institut de Recherche Informatique de Toulouse, 2015, https://hal.science/hal-03198270v1/file/Fuzzy-sets-50.pdf
Contents. 1. The emergence of fuzzy sets: A historical perspective. In: Fuzzy Logic in its 50th Year. A Perspective of New Developments, Directions and Challenges, (C. Kahraman, U. Kaymak, A. Yazici, eds.), Springer, 3-19, 2016;
2. The first steps in fuzzy set theory in France forty years ago (and before). Archives for the Philosophy and History of Soft Computing, Online journal, 2017 (1);
3. The legacy of 50 years of fuzzy sets. A discussion. Fuzzy Sets and Systems, 281, 21-31, 2015;
4. Practical methods for constructing possibility distributions. Int. J. of Intelligent Systems, 31 (3), 215-239, 2015;
5. Inconsistency management from the standpoint of possibilistic logic. Int. J. of Uncertainty, Fuzziness and Knowledge-Based Systems, 23, Suppl. 1, 2015;
6. (with D. Ciucci). Structures of opposition in fuzzy rough sets. Fundamenta Informaticae, 142 (1-4), 1-19, 2015;
7. Bridging gaps between several forms of granular computing. Granular Computing, 1, 115–126, 2016;
8. (with J-C. Bezdek). The posterity of Zadeh's 50-year-old paper. A retrospective in 101 easy pieces - and a few more. Proc. IEEE Int. Conf. on Fuzzy Systems (FUZZ-IEEE'15), Istambul, Aug. 2-5, 2015.

[2] L. A. Zadeh. "Fuzzy sets as a basis for a theory of possibility," Fuzzy Sets and Systems, 1 (1), 3-28, 1978.'

[3] D. Dubois and H. Prade. "Possibility Theory. An Approach to Computerized Processing of Uncertainty," (with the collaboration of H. Farreny, R. Martin-Clouaire, C. Testemale), Plenum Press, 1988.

[4] D. Dubois and H. Prade. "40 years of research in possibilistic logic - A survey," Proc. 34th Int. Joint Conf. on Artificial Intelligence (IJCAI'25), Montreal, Aug. 16-22, 2025.'

[5] H. Fargier, R. Sabbadin. "Qualitative decision under uncertainty: back to expected utility," Artif. Intell., 164(1-2), 245-280, 2005.

[6] G. Mauris. "Possibility distributions: A unified representation of usual direct-probability-based parameter estimation methods," Int. J. Approx. Reason., 52(9), 1232-1242, 2011.

[7] D. Dubois. "Possibility theory and statistical reasoning," Comput. Stat. Data Anal. 51(1), 47-69, 2006.

[8] K. Loquin, O. Strauss, J.-F. Crouzet. "Possibilistic signal processing: How to handle noise?," Int. J. Approx. Reason. 51(9), 1129-1144, 2010.

[9] S. Destercke, D. Dubois, E. Chojnacki: "Unifying practical uncertainty representations - I: Generalized p-boxes". Int. J. Approx. Reason. 49(3), 649-663, 2008; II: Clouds. Int. J. Approx. Reason. 49(3), 664-677, 2008.

[10] A. Levray, S. Benferhat, K. Tabia. "Possibilistic Networks: Computational Analysis of MAP and MPE Inference," Int. J. Artif. Intell. Tools, 29(3-4), 2060005:1-2060005:28, 2020.

[11] D. Dubois, H. Prade, A. Rico. "Representing qualitative capacities as families of possibility measures," Int. J. Approx. Reason. 58: 3-24 ,2015.

[12] D. Dubois, F. Faux, H. Prade, A. Rico. "Qualitative capacities: Basic notions and potential applications," Int. J. Approx. Reason. 148, 253-290, 2022.

[13] D. Dubois, H. Prade. "Possibility Theory and Its Applications: Where do we stand?," In: Handbook of Computational Intelligence, (J. Kaprzyk, W. Pedrycs, eds.), 31-60, Springer, 2015.

[14] D. Dubois and H. Prade. "Structures of opposition in possibility theory-based logic and related logics," Proc. IEEE Int. Conf. on Fuzzy Systems (FUZZ-IEEE'25), Reims, 2025.

[15] D. Dubois and H. Prade. "Reasoning and learning in the setting of possibility theory - Overview and perspectives," Int. J. Approx. Reason., 171, 109028, 2024.

[16] I. Baaj, J. P. Poli, W. Ouerdane, and N. Maudet. "Min-max inference for possibilistic rule-based system," Proc. IEEE Int. Conf. on Fuzzy Systems (FUZZ-IEEE'21), Luxembourg, Jul. 11-14, 2021.

[17] I. Baaj and P. Marquis. "Π-NeSy: A possibilistic neuro-symbolic approach," arXiv 2504.07055.753, cs.AI, 2025.

[18] .I. Baaj, D. Dubois, F. Faux, H. Prade, A. Rico, O. Strauss. "Réseau de neurones et logique: un cadre qualitatif," Rencontres francophones sur la logique floue et ses applications (LFA'22), Toulouse, 20-21 oct., 127-134, 2022.

[19] O. C. Strauss, A. Rico, J. Pasquet and L. Pibre. "Combining thresholded real values for designing an artificial neuron in a neural network," Fuzzy Sets and Systems, 2025.

[20] S. Schockaert. "Embeddings as epistemic states: Limitations on the use of pooling operators for accumulating knowledge," Int. J. Approx. Reason. 171: 108981, 2024, and arXiv:2210.05723v2, cs.AI, 2023.

From Model-Based and Adaptive Control to Evolving Fuzzy Control

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***Abstract*—Evolving fuzzy systems build and adapt fuzzy models—such as predictors and controllers—by incrementally updating their rule-base structure from data streams. On the occasion of the 60-year anniversary of fuzzy set theory, commemorated during the Fuzz-IEEE 2025 event, this brief paper revisits the historical development and core contributions of classical fuzzy and adaptive modeling and control frameworks. It then highlights the emergence and significance of evolving intelligent systems in fuzzy modeling and control, emphasizing their advantages in handling nonstationary environments. Key challenges and future directions are discussed, including safety, interpretability, and principled structural evolution.**

***Index Terms*—Fuzzy systems, adaptive control, evolving models, rule-based learning.**

I. Introduction

Research in fuzzy modeling, control, and applications has grown rapidly since Zadeh's seminal work in 1965 [1], evolving into a vast and multifaceted field. It includes key theoretical advances and a wide range of applications in engineering, industry, mechatronics, computer science, information systems, and beyond. This body of work has offered valuable opportunities to refine foundational concepts and increase the visibility of practical implementations. The established theoretical results, along with numerous successful applications—highlighting performance, smoothness, robustness, and interpretability in support of human decision-making—stand as evidence of the widespread acceptance and practical viability of fuzzy models and fuzzy control systems.

Despite evolving attitudes toward fuzzy systems, particularly within the academic community, debates have persisted between proponents and critics. As summarized by Belohlavek [2]: (i) Lindley (1987) argued that anything achievable through fuzzy logic, belief functions, upper and lower probabilities, or other alternatives to probability could be done better using probability theory; (ii) Cheeseman (1986) similarly claimed that everything achievable with fuzzy logic is better addressed probabilistically; and (iii) in 1972, Kalman—one of the pioneers of modern system modeling and control—asserted that fuzzy logic was pragmatically unconvincing due to the lack of evidence it could solve important problems.

Interestingly, Kosko [3] developed and compared a fuzzy control system with a Kalman filter-based control system for real-time target tracking. Simulation results suggested that, in many cases, fuzzy controllers may provide a robust and computationally efficient alternative to both linear and nonlinear (extended) Kalman filter approaches in real-time control—even when accurate input-output differential or difference equation models are available. Following this work—and even throughout the late 1980s, 1990s, and early 2000s—a growing number of studies confirmed similar findings, demonstrating that fuzzy control methods can offer competitive or superior performance, along with stability and robustness, in a variety of real-time, nonlinear, and uncertain environments.

In the preface of [4], Zadeh wrote: "What is not fully recognized, however, is that fuzzy control and conventional crisp control are, for the most part, complementary rather than competitive. Thus, fuzzy control is rule-based whereas conventional control is differential-equation-based; fuzzy control is task-oriented whereas conventional control is set-point-oriented; and conventional control is model-based whereas, in the case of fuzzy control, what suffices is a linguistic, rule-based description of the model. Today we see more clearly that fundamentally, conventional control is measurement-based whereas fuzzy control is perception-based. In this sense, the role model for fuzzy control is the remarkable human capability to perform a wide variety of tasks without any measurements and any computations. A canonical example of such tasks is that of driving a car in city traffic. Classical control provides no methods for automation of tasks of this type." This perspective highlights the foundational role of fuzzy sets in representing expert knowledge and supporting intelligent, flexible control strategies.

In recent years, the integration of machine learning and knowledge discovery techniques into fuzzy models and controllers has further advanced the expert-based paradigm of the 1990s. This progress has led to the development of novel and adaptive approaches that merge rule-based reasoning with data-driven learning and model-based control design. In parallel, formal tools from optimization and dynamical systems theory—such as regularization, constrained loss functions, Lyapunov stability analysis, linear matrix inequalities, H-infinity control, model predictive control, sector nonlinearity, and sum-of-squares programming—have been increasingly adopted to provide theoretical guarantees [5]–[7], further enriching fuzzy control with mathematical rigor and strengthening its alignment with contemporary control theories.

Current developments in fuzzy modeling and control have focused on strengthening their capacity to manage complexity, uncertainty, and imprecision in dynamic environments. Emphasis has shifted toward greater autonomy in model construction and adaptation, with algorithms capable of extracting and updating fuzzy rules, membership functions, and locally valid equations directly from data. These approaches significantly reduce, or even eliminate, the dependence on manual expert input, allowing more scalable and adaptive solutions aligned with current trends in modeling large datasets and data streams [7]. Furthermore, the integration of fuzzy systems with machine learning and data science has led to hybrid frameworks that support online learning, generalization, and better integration with real-world applications.

The next sections revisit the area of evolving fuzzy systems—an established branch of fuzzy modeling and control that focuses on systems capable of incrementally adapting their structure, functionality, and knowledge in response to nonstationary data streams [7]–[11]. The discussion begins with a brief overview of classical paradigms for modeling and controlling complex systems, followed by the notion of adaptive modeling and control. It then highlights the key ideas and contributions that evolving fuzzy systems have brought to the broader field over the past decade.

II. Modeling and Control of Complex Systems

The modeling and design of control systems for complex dynamical processes remain difficult and challenging tasks [12], [13]. While a rich and well-established body of theory exists for linear systems, the same level of maturity has not yet been achieved for nonlinear control. In most cases, analytic solutions to nonlinear control problems are unavailable, requiring approximate or heuristic approaches.

Conventional design methods for complex nonlinear systems are typically model-based and often laborious, involving subjective steps informed by prior experience or simulation-based tuning. A common practice is to approximate a nonlinear control law using multiple linear controllers, each designed for a specific region of the operating space. In each region, the system dynamics is assumed to be approximately linear. This partition-based (or granular) approach becomes necessary when a single linear model fails to adequately represent the system's global behavior. The nonlinear control strategy is then synthesized by switching or interpolating among the local controllers based on the current state of the system. However, this approach is inherently sensitive to modeling inaccuracies. Any significant mismatch between the model and the actual plant, or the presence of unmodeled and time-varying (evolving) dynamics, can severely degrade performance and even lead to instability. Importantly, such limitations may persist despite rigorous offline controller design.

To ensure robust performance, controllers are generally expected to tolerate a certain degree of modeling uncertainty and external disturbances [14]. However, robustness is often achieved at the expense of optimal closed-loop performance. A promising alternative lies in control systems capable of learning from experience in real time. By continuously adapting to previously unmodeled and time-varying (evolving) dynamics, these systems offer the potential to reconcile robustness with high performance—ultimately motivating the development of evolving control strategies.

A. Classical Adaptive Control and Its Limitations

Adaptive modeling and control systems adjust their parameters in response to changes in the dynamical behavior of the controlled process. The concept originated in the 1950s [15] and was soon formalized within the framework of stochastic systems by Feldbaum [16], whose work is widely regarded as the first theoretical foundation of adaptive control. Additional frameworks emerged in the early 1960s, notably the work of Mishkin and Braun [17], who provided a systematic treatment of adaptive control principles and motivated their application in engineering contexts. Around the same period, further developments and perspectives began to appear in the literature, such as the contribution by Truxal [18], reflecting the growing interest in adaptive techniques.

Adaptive modeling and control methods monitor the input/output behavior of a plant to identify, either explicitly or implicitly, the parameters of the design model and adjust the controller parameters accordingly to meet the specified performance. An adaptive system attempts to revise these parameters whenever the plant's behavior changes significantly. If the dynamical characteristics of the process vary over its operational range, the control system may be required to adapt continuously. In essence, adaptive modeling and control combine an online parameter estimator with a control law to regulate or compensate classes of plants whose parameters are either initially unknown or vary over time in unpredictable ways. The particular choices of estimator, control law, and their integration define different classes of adaptive control schemes [19].

The main approaches to adaptive control differ in whether the controller parameters are adapted directly or indirectly. In direct adaptive control, the controller parameters are adjusted directly on the basis the observed system behavior. In indirect adaptive control, the plant model parameters are first estimated online, and the controller is subsequently updated as a function of these estimates (Fig. 1).

Direct adaptive control was first introduced in the context of model reference adaptive systems for aircraft control [20], while indirect adaptive control emerged in digital process control applications [21]. These adaptation strategies underpin classical schemes such as Model Reference Adaptive Control (MRAC) and Self-Tuning Regulators (STR) [22], [23], which remain central to online adaptive control. MRAC defines the desired closed-loop behavior via an explicit reference model and adapts the controller to minimize the model-following error, whereas STR estimates plant parameters online and recomputes the controller using techniques such as pole placement or optimal control synthesis.

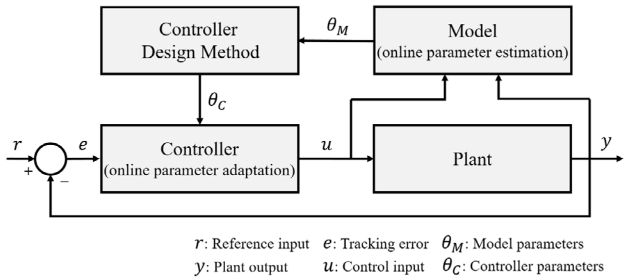

Fig. 1. Indirect adaptive control scheme

The use of adaptive control is based on the assumption that, for any possible values of the coefficients of the plant model, there exists a controller—with fixed structure and complexity—capable of meeting the design specifications through appropriate tuning of its parameters [24]. In this context, the task of adaptation is to determine suitable values for the controller parameters. It is worth noting that conventional controller design relies on an offline mathematical model of the process. Once an adequate model is obtained, established design methods are used to synthesize a controller that meets the required performance specifications. In contrast to this static procedure, adaptive control systems aim to adjust model and controller parameters online, while keeping their structures fixed by design. These adjustments are driven by real-time input-output data from the process.

Ultimately, in conventional design, both the model and controller structures—as well as their coefficients and gains—are fixed during operation. Adaptive systems allow real-time adjustment of model coefficients and controller gains but still assume fixed structures. The simultaneous adaptation of both the structure and parameters of the model and controller remains an open challenge. As highlighted by Annaswamy and Fradkov [25], adaptive control remains primarily concerned with parameter adaptation—that is, the tuning of model coefficients and controller gains. This limitation has motivated the development of evolving control schemes capable of structural and parametric learning from data streams.

B. Toward Evolving Modeling and Fuzzy Control

Evolving fuzzy systems represent a major shift in the field of adaptation, learning, and self-organizing systems, with impact extending well beyond fuzzy modeling and control, including online classification, clustering, forecasting, and decision-making in dynamic environments [7]. In contrast to conventional modeling and control methods—which require an offline design or training phase—evolving fuzzy systems simultaneously adapt both their structure and parameters (Fig. 2). They emphasize incremental learning and self-development, can operate entirely online, start from scratch without prior process knowledge, and allow human knowledge to be incorporated at any stage. The general scheme illustrated in Fig. 2 is applicable to both evolving plant models and controllers, provided that the appropriate input-output signals are considered. A practical example employing both an evolving Takagi–Sugeno [26] fuzzy model of the process and an evolving fuzzy controller, within a parallel distributed compensation (PDC) [5] strategy, was first presented in [6], where bounded control inputs and Lyapunov stability were formally ensured.

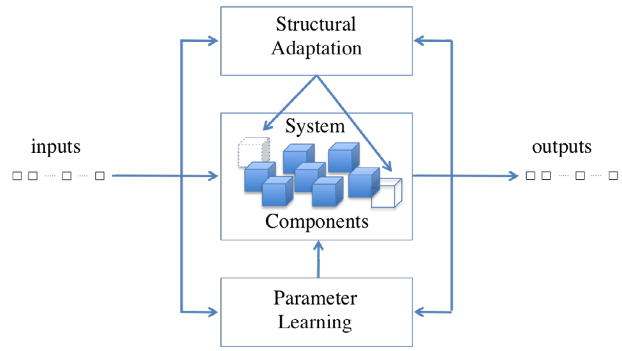

Fig. 2. Evolving system—plant model or controller: Online adaptation of both structure and parameters

Quoting the pioneers of the area [27]: "The newly emerging concept of dynamically evolving structures, which was, to some extent, already applied to neural networks [28], brought a powerful new concept of evolving fuzzy systems (EFS). EFS combine: (i) the interpolation abilities of fuzzy systems; (ii) their flexibility; (iii) the linguistic interpretability of fuzzy systems; and (iv) the adaptive feature of online learning techniques. This new topic was introduced during the last decade [8]–[10], and quickly numerous applications to problems of modeling, control, prediction, classification, and data processing in a dynamically changing and evolving environment were also developed, including some successful industrial applications [29]."

The terms adaptation, learning system, and self-organization often appear in the literature as synonymous. Early work views an adaptive system as one that is insensitive to changes in its environment, that is, one that performs acceptably well over a range of inputs [30]. In contrast, a learning system is one that operates satisfactorily under changing environmental conditions in which an adaptive system does not improve its performance [31]. Self-organization refers to a process in which the structure and behavior of a system emerge from interactions among its components. This contrasts with conventional modeling and control systems, which rely on centralized or fixed hierarchical structures. Self-organization enables more flexible, adaptable, and resilient systems capable of shaping their own behavior [32].

Current data-driven learning models and control methods typically require an offline training phase, often based on input-output datasets generated by applying a wide range of input conditions in closed loop. This process is frequently infeasible, especially for originally unstable systems. Moreover, these methods lack the capability to operate from scratch and to perform concurrent, online adjustment of both system structure and parameters—a key requirement for fully self-organizing and evolving systems.

C. Open Challenges and Paths Forward

Closing the loop with a controller that evolves from scratch is risky, especially in open-loop unstable systems or those involving safety concerns. While evolving fuzzy systems offer the potential to self-develop from minimal prior knowledge, their effective use in practice requires guarantees of safe operation—especially during the early stages of learning. To this end, possible strategies include: (i) using initial conservative controllers (fallback control) while the evolving model and controller learn cooperatively through observation; (ii) incorporating human-in-the-loop supervision; (iii) restricting exploration to known safe regions of the input space; and (iv) implementing bounded output policies that ensure safe actuator behavior. Evolving fuzzy control should balance autonomy with safety-aware initialization and updates to maintain system stability and operational integrity.

Beyond safety, interpretability remains a key requirement for the practical adoption of evolving fuzzy controllers. Future research should address how to evolve models and controllers in a controlled manner. Structural changes—such as the addition of new fuzzy rules—should be validated in shadow or standby mode via separate computational simulations, or be subject to human review before activation. Risk metrics and domain-specific safety constraints should guide the evolution process to protect critical state variables. Additionally, robustness to uncertainty and resilience to anomalies are essential to prevent overreaction and maintain system stability.

III. Conclusion

This paper revisited the fundamental concepts of adaptive modeling and control, highlighting their capabilities and limitations in the face of nonstationary and complex environments. While conventional adaptive systems support online parameter adjustment, they operate under fixed model and controller structures, which limits their ability to handle complex, evolving processes. Evolving fuzzy systems fill this gap by supporting both parametric and structural adaptation from data streams, often without requiring prior process knowledge. Their capacity for incremental learning and integration of expert opinion makes them a promising path forward. However, applying these systems in real-world scenarios—particularly those inherently unstable or involving safety risks—demands additional mechanisms to ensure safe learning, interpretability, and resilience. As such, future efforts should balance structural adaptability and domain-aware validation.

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References

[1] L. A. Zadeh. Fuzzy sets. Information and Control, 8(3), 338–353, 1965.
[2] R. Belohlavek, J. Dauben, J. Klir, Fuzzy Logic and Mathematics: A Historical Perspective. New York, NY: Oxford, 2017.
[3] B. Kosko, Fuzzy Logic: A Dynamical System Approach to Machine Intelligence, Englewood Cliffs, NJ: Prentice-Hall, 1992.
[4] H. Ying, Fuzzy Control and Modeling: Analytical Foundations and Applications. New York, NY: IEEE Press, 2000.
[5] K. Tanaka, H. O. Wang, Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach. J. Wiley & Sons, Hoboken, 2004.
[6] D. Leite, R. Palhares, V. Campos, F. Gomide, Evolving granular fuzzy model-based control of nonlinear dynamic systems. IEEE Transactions on Fuzzy Systems, 23(4), 923-938, 2015.
[7] I. Škrjanc, J. A. Iglesias, A. Sanchis, D. Leite, E. Lughofer, F. Gomide, Evolving fuzzy and neuro-fuzzy approaches in clustering, regression, identification, and classification: A survey. Information Sciences, 490, 344-368, 2019.
[8] N. Kasabov, Q. Song, DENFIS: Dynamic evolving neural-fuzzy inference system and its application for time-series prediction. IEEE Transactions on Fuzzy Systems, 10, 144–154, 2002.
[9] P. Angelov, Evolving Rule-based Models: A Tool for Design of Flexible Adaptive Systems. Heidelberg: Springer, 2002.
[10] P. Angelov, D. Filev, An approach to online identification of Takagi–Sugeno fuzzy models. IEEE Transactions on Systems, Man, and Cybernetics B, 34, 484–498, 2004.
[11] M. Sayed-Mouchaweh, E. Lughofer (Eds.), Learning in Non-Stationary Environments: Methods and Applications. Springer, New York, 2012.
[12] H. Khalil, Nonlinear Systems. Upper Saddle River, NJ: Prentice Hall, 2002.
[13] D. A. Copp, J. P. Hespanha, Simultaneous nonlinear model predictive control and state estimation. Automatica, 77, 143–154, 2017.
[14] M. Bernal, A. Sala, Z. Lendek, T. Guerra, Analysis and Synthesis of Nonlinear Control Systems. Cham, CH: Springer, 2022.
[15] A. Aseltine, R. Mancini, C. Saturne, A survey of adaptive control systems. IRE Transactions on Automatic Control, 6, 102–108, 1958.
[16] A. Feldbaum, Dual control theory. Automation and Remote Control, 21, 1240–1249, 1960.
[17] E. Mishkin, L. Brown, Adaptive Control Systems. New York, NY: McGraw-Hill, 1961.
[18] J. Truxal, Adaptive control. Proceedings of IFAC, 1, 386–392, 1963.
[19] P. Ioannou, B. Fidan, Adaptive Control Tutorial. Philadelphia, PA: SIAM, 2006.
[20] H. Whitaker, J. Yamron, A. Kezer, Design of a model-reference adaptive control system for aircraft. Tech. Report R-164, MIT Instrumentation Laboratory, Cambridge, MA, 1958.
[21] R. Kalman, Design of self-optimizing control systems. Transactions of ASME Journal of Basic Engineering, 80, 468–478, 1958.
[22] S. Blažič, I. Škrjanc, D. Matko, Globally stable direct fuzzy model reference adaptive control. Fuzzy Sets and Systems, 139(1), 3–33, 2003.
[23] I. Škrjanc, S. Blažič, D. Matko, Model-reference fuzzy adaptive control as a framework for nonlinear system control. Journal of Intelligent & Robotic Systems, 36(3), 331–347, 2003.
[24] I. Landau, R. Lozano, M. Saad, A. Karimi, Adaptive Control: Algorithms, Analysis and Applications. New York, NY: Springer, 2011.
[25] A. Annaswamy, A. Fradkov, A historical perspective of adaptive control and learning. https://doi.org/10.48550/arXiv.2108.11336, 2022.
[26] T. Takagi, M. Sugeno, Fuzzy identification of systems and its applications to modeling and control. IEEE Transactions on Systems, Man, and Cybernetics, 15(1), 116–132, 1985.
[27] P. Angelov, N. Kasabov, D. Filev, Guest editorial evolving fuzzy systems: Preface to the special section. IEEE Transactions on Fuzzy Systems, 16, 1391–1392, 2008.
[28] B. Fritzke, Growing cell structures—A self-organizing network for unsupervised and supervised learning. Neural Netw, 7, 1441–1460, 1994.
[29] A. Kordon, Inferential sensors as potential application area of intelligent evolving systems. Proceedings of IEEE International Symposium on Evolving Fuzzy Systems, UK, 2006.
[30] L. Zadeh, On the definition of adaptivity. Proceedings of the IEEE, 51, 469–470, 1963.
[31] J. Hill, G. McMurthy, K. Fu, A computer-simulated on-line experiment in learning control systems. Simulation, 4, 104–116, 1965.
[32] G. Saridis, Self-organizing control and applications to trainable manipulators and learning prostheses. IFAC Proceedings, 8, 632–640, 1975.

Fuzzy Machine Learning – Celebration of 60 years of fuzzy sets

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***Abstract*—We consider *Fuzzy Machine Learning* (FML) as the most significant recent contribution in the fuzzy systems area in the past 10 years. Machine learning draws its power from various disciplines, including computer science, cognitive science, and statistics. Although machine learning has achieved great advancements in both theory and practice, its methods have some limitations when dealing with complex situations and highly uncertain environments. Insufficient data, imprecise observations, and ambiguous information/relationships can all confound traditional machine learning systems. To address these problems, researchers have integrate different fuzzy techniques into machine leaning called FML as a solution. FML are divided into five categories: (a) fuzzy classical machine learning; (b) fuzzy transfer learning; (c) fuzzy data stream learning; (d) fuzzy reinforcement learning; and (e) fuzzy recommender systems. This paper should provide researchers with a brief understanding of the current progress in FML.**

I. Introduction

FML stands out as an invaluable ally in the realm of complex, and dynamic (uncertain) environments, presenting substantial advantages that elevate its efficacy. Unlike traditional machine learning approaches, fuzzy techniques that generally based on the concept of fuzzy sets and fuzzy theory excel in capturing and navigating the nuanced shades of uncertainty inherent in dynamic scenarios. Their inherent ability to model uncertainty empowers it to gracefully adapt to the ever-changing patterns that characterize dynamic environments. In situations where traditional models might falter or struggle to keep pace, fuzzy techniques emerge as robust problem-solvers, providing a more accurate representation of the inherent fuzziness present in real-world data. Furthermore, in the relentless quest for interpretability, fuzzy machine learning triumphs. Its models not only navigate complexity but also offer clear insights into decision-making processes. This interpretability proves to be a critical asset in dynamic environments where understanding the rationale behind model decisions is paramount.

In the past decade, there have been over $500,000$ articles in high-quality journals and conference proceedings related to FML. Recently, we published a paper [1] in IEEE Transactions on Fuzzy Systems to provide a comprehensively survey that summarizes the developments and achievements in the field of FML. In this paper, we divide FML into five groups (see Fig. 1) : (a) fuzzy classical machine learning; (b) fuzzy transfer learning; (c) fuzzy data stream learning; (d) fuzzy reinforcement learning; and (e) fuzzy recommender systems.

TABLE I
SOME SOTA WORKS RELATED TO FUZZY CLASSICAL MACHINE LEARNING.

Fuzzy technique	Non-deep learning	Deep learning algorithm	
		CNN	Others(eg., RNN)
Fuzzy clustering	[2] [3]	[4] [5]	[6] [7]
Type-1 fuzzy systems	[8] [9]	[10] [11]	[12] [13]
Type-2 fuzzy systems	[14]	[15] [16]	[17] [18]

TABLE II
SUMMARY OF THE SOTA PAPERS IN FUZZY TRANSFER LEARNING.

Fuzzy techniques	Type	
	Regression	Classification
Fuzzy sets	[20] [21]	[22] [23]
Fuzzy systems	[24] [25] [26]	[27] [28] [29] [30] [31]
Fuzzy relations	-	[32]

A. Fuzzy classical machine learning

Classical machine learning algorithms, such as decision trees, support vector machines, and neural networks, have been responsible for remarkable achievements both theoretically and from a practical point of view. Numerous articles involve combining fuzzy techniques with classical machine learning algorithms to overcome different types of problems with uncertainty, such as incomplete information and imprecise observations. Table I provides some SOTA works related to fuzzy classical machine learning.

B. Fuzzy transfer learning

Transfer learning [19] tries to train a well-performed model in one domain (target) by leveraging knowledge from another domain (source) that has different distribution or learning tasks compared with the previous one. Notably, most current transfer learning methods have limitations when handling real-world situations with uncertainty, such as when only a few labeled instances are available. To overcome these problems, many researchers have turned to fuzzy sets and fuzzy logic. We have divided the summary of recent works into three areas based on the fuzzy technique used. These are fuzzy sets, fuzzy systems, and fuzzy relations. Table II summarizes recent achievements in the field of fuzzy transfer learning.

C. Fuzzy data stream learning

Learning from data streams [33] involves developing algorithms and techniques to adaptively and incrementally process

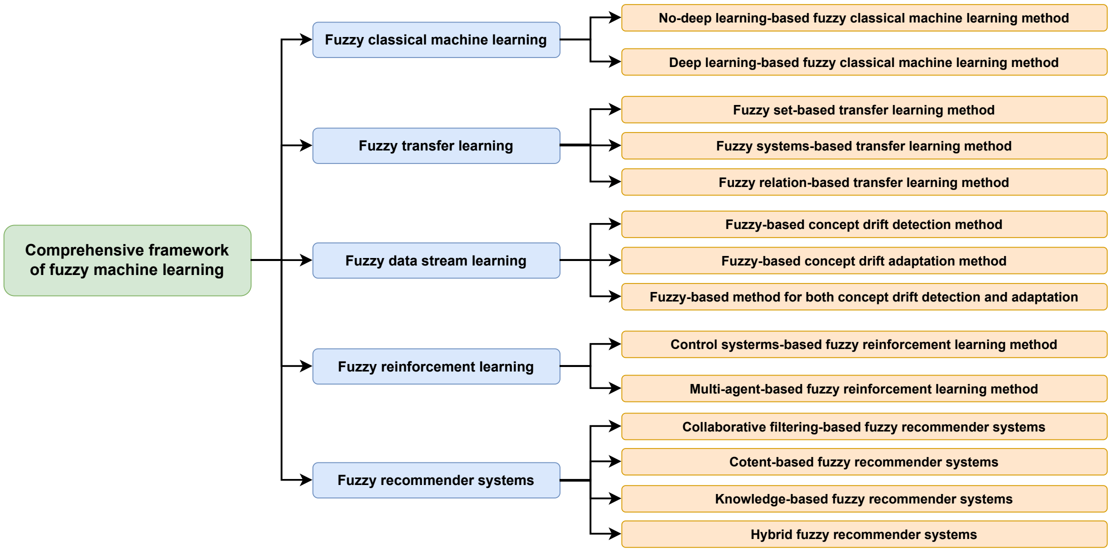

Fig. 1. Comprehensive framework of fuzzy machine learning.

TABLE III
SUMMARY OF THE SOTA ACHIEVEMENTS IN FUZZY DATA STREAM LEARNING.

Fuzzy techniques	Type			
	Detection	Adaptation	Detection and adaptation	Others
Fuzzy clustering	[34] [35]	[36] [37]	-	-
Fuzzy set theory	[38] [39]	-	-	-
Fuzzy systems	-	[40]	[41] [42]	[43] [44] [45]
Fuzzy time series	-	[46] [47]	-	-
Others	[48]	[49] [50]	[51] [52]	[53]

and learn from continuously arriving data. Unlike traditional machine learning scenarios where a static dataset is available for offline training, data stream learning deals with dynamic, evolving data streams that may not be stored entirely. However, data streams often exhibit concept drift, which refers to changes in the statistical properties of the data. Detecting and adapting to concept drift are two important challenges in data stream learning. One approach is to continuously monitor the data and update models or retrain them periodically to account for changes. Another approach is to use online learning techniques that can adapt to changes in the data stream in real-time. While concept drift often come with some uncertainty problems – for example, making predictions from data streams with mixed drift problems and detecting drift in data streams with missing values – researchers are considering the application of fuzzy techniques to address these challenges. Table III summarizes some recent achievements in the field of fuzzy data stream learning.

D. Fuzzy reinforcement learning

Reinforcement learning (RL) [54] represents a powerful paradigm in machine learning, where agents learn to make decisions through interaction with an environment, guided by a system of rewards or penalties. However, the traditional RL framework is not without its challenges, especially in scenarios where the training process is inherently slow due to complex and uncertain environments or sparse reward signals. Fuzzy RL emerges as a promising approach to address these limitations, leveraging fuzzy logic to enhance training efficiency and overcome the hurdles associated with slow reinforcement processes. Table IV summarizes some recent achievements in the field of fuzzy RL.

TABLE IV
SUMMARY OF THE SOTA ACHIEVEMENTS IN FUZZY REINFORCEMENT LEARNING.

Fuzzy techniques	Type		
	Control systems	Multi-agent RL	Others
Fuzzy systems	[55] [56] [57]	[58] [59]	[60] [61]
Others	[62]	-	[63]

E. Fuzzy recommender system

In real-world recommender systems, descriptions of user preferences and item features, item values, and business knowledge are often vague, imprecise, and plagued with uncertainty. And, further, these issues can occur across the entire recommendation process from collecting the data to generating the recommendations. Other key problems that can occur with recommender systems include sparsely populated user-item matrices and problems with measuring the similarity of items and users. Commonly used fuzzy techniques to deal with these issues include intuitionistic fuzzy sets, fuzzy user profiles, fuzzy rule-based systems, and fuzzy similarity. Table V summarizes some recent works in the field of fuzzy-based recommender systems.

II. CONCLUSION

Our survey paper shows that fuzzy techniques can significantly improve machine learning algorithms by providing

TABLE V

SUMMARY OF THE SOTA FUZZY TECHNIQUES-BASED RECOMMENDER SYSTEMS ACIEVEMENTS.

Fuzzy techniques	Type			
	Collaborative	Content-Based	Knowledge-Based	Hybrid
Fuzzy systems	[64]	[65]	[66] [67]	[68] [69]
Intuitionistic fuzzy set	[70]	-	-	-
Fuzzy clustering	[71] [72]	-	-	-
Fuzzy profile	[73]	-	-	-
Others	[74]	[75]	[76] [77]	[78]

a way to handle different uncertainty situations. The main improvements are reflected in the following five aspects: 1) enhancing the representation of the inputs; 2) improving the learning process of different machine learning algorithms; 3) enhancing measurement accuracy and reliability; 4) improving the accuracy of the matching function; 5) enhancing the performance (e.g., accuracy, robustness and interpretability) of the output results. In future research, several new directions in the field of FML warrant thorough consideration. For instance, applying fuzzy techniques to address open-set transfer learning problems. In addition, multi-stream learning, multi-agent RL, and cross-domain recommendations are three challenge problems that are far from being solved.

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REFERENCES

1 Lu, J., Ma, G., and Zhang, G., "Fuzzy machine learning: A comprehensive framework and systematic review," *IEEE Transactions on Fuzzy Systems*, vol. 32, no. 7, pp. 3861–3878, 2024.

2 Mishro, P. K., Agrawal, S., Panda, R., and Abraham, A., "A novel type-2 fuzzy c-means clustering for brain mr image segmentation," *IEEE Transactions on Cybernetics*, vol. 51, no. 8, pp. 3901–3912, 2020.

3 Jiao, L., Yang, H., Liu, Z., and Pan, Q., "Interpretable fuzzy clustering using unsupervised fuzzy decision trees," *Information Sciences*, vol. 611, pp. 540–563, 2022.

4 Sideratos, G., Ikonomopoulos, A., and Hatziargyriou, N. D., "A novel fuzzy-based ensemble model for load forecasting using hybrid deep neural networks," *Electric Power Systems Research*, vol. 178, p. 106025, 2020.

5 Tan, D., Huang, Z., Peng, X., Zhong, W., and Mahalec, V., "Deep adaptive fuzzy clustering for evolutionary unsupervised representation learning," *IEEE Transactions on Neural Networks and Learning Systems*, 2023.

6 Yu, J., Kim, J., Li, X., Jong, Y., Kim, K., and Ryang, G., "Water quality forecasting based on data decomposition, fuzzy clustering and deep learning neural network," *Environmental Pollution*, vol. 303, p. 119136, 2022.

7 Wang, Y., Ishibuchi, H., Er, M. J., and Zhu, J., "Unsupervised multilayer fuzzy neural networks for image clustering," *Information Sciences*, vol. 622, pp. 682–709, 2023.

8 Xia, H., Tang, J., Yu, W., Cui, C., and Qiao, J., "Takagi–sugeno fuzzy regression trees with application to complex industrial modeling," *IEEE Transactions on Fuzzy Systems*, 2022.

9 Ma, G., Lu, J., Liu, F., Fang, Z., and Zhang, G., "Multiclass classification with fuzzy-feature observations: Theory and algorithms," *IEEE Transactions on Cybernetics*, vol. 54, no. 2, pp. 1048–1061, 2022.

10 Kumar, A., Sangwan, S. R., Arora, A., and Menon, V. G., "Depress-dcnf: a deep convolutional neuro-fuzzy model for detection of depression episodes using iomt," *Applied Soft Computing*, vol. 122, p. 108863, 2022.

11 Zhou, W., Liu, M., and Xu, Z., "The dual-fuzzy convolutional neural network to deal with handwritten image recognition," *IEEE Transactions on Fuzzy Systems*, vol. 30, no. 12, pp. 5225–5236, 2022.

12 Nasiri, H. and Ebadzadeh, M. M., "Mfrfnn: Multi-functional recurrent fuzzy neural network for chaotic time series prediction," *Neurocomputing*, vol. 507, pp. 292–310, 2022.

13 Pham, P., Nguyen, L. T., Nguyen, N. T., Kozma, R., and Vo, B., "A hierarchical fused fuzzy deep neural network with heterogeneous network embedding for recommendation," *Information Sciences*, vol. 620, pp. 105–124, 2023.

14 Zadeh, L. A., "The concept of a linguistic variable and its application to approximate reasoning—i," *Information Sciences*, vol. 8, no. 3, pp. 199–249, 1975.

15 Shen, T., Wang, J., Gou, C., and Wang, F., "Hierarchical fused model with deep learning and type-2 fuzzy learning for breast cancer diagnosis," *IEEE Transactions on Fuzzy Systems*, vol. 28, no. 12, pp. 3204–3218, 2020.

16 Sharma, T. and Verma, N. K., "Estimating depth and global atmospheric light for image dehazing using type-2 fuzzy approach," *IEEE Transactions on Emerging Topics in Computational Intelligence*, vol. 6, no. 1, pp. 93–102, 2020.

17 Wang, Y. and Luo, C., "An intelligent quantitative trading system based on intuitionistic-gru fuzzy neural networks," *Applied Soft Computing*, vol. 108, p. 107471, 2021.

18 Qazani, M. R. C., Asadi, H., Al-Ashmori, M., Mohamed, S., Lim, C. P., and Nahavandi, S., "Time series prediction of driving motion scenarios using fuzzy neural networks:* motion signal prediction using fnns," in *2021 IEEE International Conference on Mechatronics*, 2021, pp. 1–6.

19 Pan, S. J. and Yang, Q., "A survey on transfer learning," *IEEE Transactions on Knowledge and Data Engineering*, vol. 22, no. 10, pp. 1345–1359, 2010.

20 Shukla, A. K., Kumar, S., Jagdev, R., Muhuri, P. K., and Lohani, Q. D., "Interval type-2 fuzzy weighted extreme learning machine for gdp prediction," in *2018 International Joint Conference on Neural Networks*, 2018, pp. 1–8.

21 Kumar, S., Shukla, A. K., Muhuri, P. K., and Lohani, Q. D., "Transfer learning based gdp prediction from uncertain carbon emission data," in *2019 IEEE International Conference on Fuzzy Systems*, 2019, pp. 1–6.

22 Behbood, V., Lu, J., Zhang, G., and Pedrycz, W., "Multistep fuzzy bridged refinement domain adaptation algorithm and its application to bank failure prediction," *IEEE Transactions on Fuzzy Systems*, vol. 23, no. 6, pp. 1917–1935, 2015.

23 Ma, G., Lu, J., and Zhang, G., "Multi-source domain adaptation with interval-valued target data via fuzzy neural networks," *IEEE Transactions on Fuzzy Systems*, 2024.

24 Zuo, H., Zhang, G., Pedrycz, W., Behbood, V., and Lu, J., "Granular fuzzy regression domain adaptation in takagi–sugeno fuzzy models," *IEEE Transactions on Fuzzy Systems*, vol. 26, no. 2, pp. 847–858, 2018.

25 Lu, J., Zuo, H., and Zhang, G., "Fuzzy multiple-source transfer learning," *IEEE Transactions on Fuzzy Systems*, vol. 28, no. 12, pp. 3418–3431, 2020.

26 Che, X., Zuo, H., Lu, J., and Chen, D., "Fuzzy multioutput transfer learning for regression," *IEEE Transactions on Fuzzy Systems*, vol. 30, no. 7, pp. 2438–2451, 2021.

27 Meher, S. K. and Kothari, N. S., "Interpretable rule-based fuzzy elm and domain adaptation for remote sensing image classification," *IEEE Transactions on Geoscience and Remote Sensing*, 2020.

28 Deng, Z., Xu, P., Xie, L., Choi, K., and Wang, S., "Transductive joint-knowledge-transfer tsk fs for recognition of epileptic eeg signals," *IEEE Transactions on Neural Systems and Rehabilitation Engineering*, vol. 26, no. 8, pp. 1481–1494, 2018.

29 Xu, P., Deng, Z., Wang, J., Zhang, Q., Choi, K., and Wang, S., "Transfer representation learning with tsk fuzzy system," *IEEE Transactions on Fuzzy Systems*, 2019.

30 Li, K., Lu, J., Zuo, H., and Zhang, G., "Source-free multi-domain adaptation with fuzzy rule-based deep neural networks," *IEEE Transactions on Fuzzy Systems*, 2023.

31 Ma, G., Lu, J., Liu, F., Fang, Z., and Zhang, G., "Domain adaptation with interval-valued observations: Theory and algorithms," *IEEE Transactions on Fuzzy Systems*, 2024.

32 Liu, Feng and Zhang, Guangquan and Lu, Jie, "Multi-source heterogeneous unsupervised domain adaptation via fuzzy-relation neural networks," *IEEE Transactions on Fuzzy Systems*, 2020.

33 Sun, Y., Tang, K., Minku, L. L., Wang, S., and Yao, X., "Online ensemble learning of data streams with gradually evolved classes," *IEEE Transactions on Knowledge and Data Engineering*, vol. 28, no. 6, pp. 1532–1545, 2016.

34 Zhang, B., Qin, S., Wang, W., Wang, D., and Xue, L., "Data stream clustering based on fuzzy c-mean algorithm and entropy theory," *Signal Processing*, vol. 126, pp. 111–116, 2016.
35 Hong, T., Chen, C., Li, Y., and Wu, M., "Using fuzzy c-means to discover concept-drift patterns for membership functions," *Transactions on Fuzzy Sets and Systems*, vol. 1, no. 2, pp. 21–31, 2022.
36 Song, Y., Lu, J., Lu, H., and Zhang, G., "Fuzzy clustering-based adaptive regression for drifting data streams," *IEEE Transactions on Fuzzy Systems*, vol. 28, no. 3, pp. 544–557, 2020.
37 Bechini, A., Marcelloni, F., and Renda, A., "Tsf-dbscan: a novel fuzzy density-based approach for clustering unbounded data streams," *IEEE Transactions on Fuzzy Systems*, vol. 30, no. 3, pp. 623–637, 2022.
38 Zhang, Q., Wu, D., Zhang, G., and Lu, J., "Fuzzy user-interest drift detection based recommender systems," in *2016 IEEE International Conference on Fuzzy Systems*, 2016, pp. 1274–1281.
39 Dong, F., Zhang, G., Lu, J., and Li, K., "Fuzzy competence model drift detection for data-driven decision support systems," *Knowledge-Based Systems*, vol. 143, pp. 284–294, 2018.
40 García-Vico, Á. M., Carmona, C. J., Gonzalez, P., Seker, H., and del Jesus, M. J., "Fepds: A proposal for the extraction of fuzzy emerging patterns in data streams," *IEEE Transactions on Fuzzy Systems*, vol. 28, no. 12, pp. 3193–3203, 2020.
41 Lughofer, E. and Angelov, P., "Handling drifts and shifts in on-line data streams with evolving fuzzy systems," *Applied Soft Computing*, vol. 11, no. 2, pp. 2057–2068, 2011.
42 Yu, H., Lu, J., and Zhang, G., "Topology learning-based fuzzy random neural networks for streaming data regression," *IEEE Transactions on Fuzzy Systems*, vol. 30, no. 2, pp. 412–425, 2020.
43 Gu, X., "An explainable semi-supervised self-organizing fuzzy inference system for streaming data classification," *Information Sciences*, vol. 583, pp. 364–385, 2022.
44 Ferdaus, M. M., Pratama, M., Anavatti, S. G., and Garratt, M. A., "Palm: An incremental construction of hyperplanes for data stream regression," *IEEE Transactions on Fuzzy Systems*, vol. 27, no. 11, pp. 2115–2129, 2019.
45 Liu, Y., Zhao, J., Wang, W., and Pedrycz, W., "Prediction intervals for granular data streams based on evolving type-2 fuzzy granular neural network dynamic ensemble," *IEEE Transactions on Fuzzy Systems*, vol. 29, no. 4, pp. 874–888, 2020.
46 e Silva, P. C. d. L., Junior, C. A. S., Alves, M. A., Silva, R., Cohen, M. W., and Guimarães, F. G., "Forecasting in non-stationary environments with fuzzy time series," *Applied Soft Computing*, vol. 97, p. 106825, 2020.
47 Severiano, C. A., e Silva, P. C. d. L., Cohen, M. W., and Guimarães, F. G., "Evolving fuzzy time series for spatio-temporal forecasting in renewable energy systems," *Renewable Energy*, vol. 171, pp. 764–783, 2021.
48 Liu, A., Lu, J., and Zhang, G., "Concept drift detection: dealing with missing values via fuzzy distance estimations," *IEEE Transactions on Fuzzy Systems*, vol. 29, no. 11, pp. 3219–3233, 2020.
49 Liu, A., Zhang, G., and Lu, J., "Fuzzy time windowing for gradual concept drift adaptation," in *2017 IEEE International Conference on Fuzzy Systems*, 2017, pp. 1–6.
50 Song, Y., Zhang, G., Lu, H., and Lu, J., "A fuzzy drift correlation matrix for multiple data stream regression," in *2020 IEEE International Conference on Fuzzy Systems*, 2020, pp. 1–6.
51 Dong, F., Lu, J., Zhang, G., and Li, K., "Active fuzzy weighting ensemble for dealing with concept drift," *International Journal of Computational Intelligence Systems*, vol. 11, no. 1, pp. 438–450, 2018.
52 Zhang, N., Yao, X., and Luo, C., "The prediction of online time series with concept drift based on dynamic intuitionistic fuzzy cognitive map," *Intelligent Data Analysis*, vol. 25, no. 4, pp. 949–972, 2021.
53 Yu, H., Lu, J., Liu, A., Wang, B., Li, R., and Zhang, G., "Real-time prediction system of train carriage load based on multi-stream fuzzy learning," *IEEE Transactions on Intelligent Transportation Systems*, vol. 23, no. 9, pp. 15 155–15 165, 2022.
54 Sutton, R. S. and Barto, A. G., *Reinforcement learning: An introduction*. MIT press, 2018.
55 Zhang, H., Zhang, K., Cai, Y., and Han, J., "Adaptive fuzzy fault-tolerant tracking control for partially unknown systems with actuator faults via integral reinforcement learning method," *IEEE Transactions on Fuzzy Systems*, vol. 27, no. 10, pp. 1986–1998, 2019.
56 Zhang, K., Zhang, H., Mu, Y., and Liu, C., "Decentralized tracking optimization control for partially unknown fuzzy interconnected systems via reinforcement learning method," *IEEE Transactions on Fuzzy Systems*, vol. 29, no. 4, pp. 917–926, 2020.
57 Kumar, N., Rahman, S. S., and Dhakad, N., "Fuzzy inference enabled deep reinforcement learning-based traffic light control for intelligent transportation system," *IEEE Transactions on Intelligent Transportation Systems*, vol. 22, no. 8, pp. 4919–4928, 2020.
58 Li, J., Shi, H., and Hwang, K., "Using fuzzy logic to learn abstract policies in large-scale multiagent reinforcement learning," *IEEE Transactions on Fuzzy Systems*, vol. 30, no. 12, pp. 5211–5224, 2022.
59 Fang, B., Zheng, C., Wang, H., and Yu, T., "Two stream fused fuzzy deep neural network for multi-agent learning," *IEEE Transactions on Fuzzy Systems*, 2022.
60 Fathinezhad, F., Derhami, V., and Rezaeian, M., "Supervised fuzzy reinforcement learning for robot navigation," *Applied Soft Computing*, vol. 40, pp. 33–41, 2016.
61 Hein, D., Hentschel, A., Runkler, T., and Udluft, S., "Particle swarm optimization for generating interpretable fuzzy reinforcement learning policies," *Engineering Applications of Artificial Intelligence*, vol. 65, pp. 87–98, 2017.
62 Yin, L. and Li, Y., "Fuzzy vector reinforcement learning algorithm for generation control of power systems considering flywheel energy storage," *Applied Soft Computing*, vol. 125, p. 109149, 2022.
63 Shi, H., Lin, Z., Zhang, S., Li, X., and Hwang, K., "An adaptive decision-making method with fuzzy bayesian reinforcement learning for robot soccer," *Information Sciences*, vol. 436, pp. 268–281, 2018.
64 Kuanr, M., Rath, B. K., and Mohanty, S. N., "Crop recommender system for the farmers using mamdani fuzzy inference model," *International Journal of Engineering & Technology*, vol. 7, no. 4.15, pp. 277–280, 2018.
65 Rutkowski, T., Romanowski, J., Woldan, P., Staszewski, P., Nielek, R., and Rutkowski, L., "A content-based recommendation system using neuro-fuzzy approach," in *2018 IEEE International Conference on Fuzzy Systems*, 2018, pp. 1–8.
66 Sulthana, A. R. and Ramasamy, S., "Ontology and context based recommendation system using neuro-fuzzy classification," *Computers & Electrical Engineering*, vol. 74, pp. 498–510, 2019.
67 Yan, S., Pirooznia, S., Heidari, A., Navimipour, N. J., and Unal, M., "Implementation of a product-recommender system in an iot-based smart shopping using fuzzy logic and apriori algorithm," *IEEE Transactions on Engineering Management*, 2022.
68 Kermany, N. R. and Alizadeh, S. H., "A hybrid multi-criteria recommender system using ontology and neuro-fuzzy techniques," *Electronic Commerce Research and Applications*, vol. 21, pp. 50–64, 2017.
69 Manogaran, G., Varatharajan, R., and Priyan, M. K., "Hybrid recommendation system for heart disease diagnosis based on multiple kernel learning with adaptive neuro-fuzzy inference system," *Multimedia tools and applications*, vol. 77, pp. 4379–4399, 2018.
70 Guo, J., Deng, J., and Wang, Y., "An intuitionistic fuzzy set based hybrid similarity model for recommender system," *Expert Systems with Applications*, vol. 135, pp. 153–163, 2019.
71 Katarya, R. and Verma, O., "Recommender system with grey wolf optimizer and fcm," *Neural Computing and Applications*, vol. 30, pp. 1679–1687, 2018.
72 Selvi, C. and Sivasankar, E., "A novel optimization algorithm for recommender system using modified fuzzy c-means clustering approach," *Soft Computing*, vol. 23, pp. 1901–1916, 2019.
73 Wang, P., Wang, Y., Zhang, L. Y., and Zhu, H., "An effective and efficient fuzzy approach for managing natural noise in recommender systems," *Information Sciences*, vol. 570, pp. 623–637, 2021.
74 Saravanan, B., Mohanraj, V., and Senthilkumar, J., "A fuzzy entropy technique for dimensionality reduction in recommender systems using deep learning," *Soft Computing*, vol. 23, pp. 2575–2583, 2019.
75 Yera, R., Alzahrani, A. A., and Martínez, L., "A fuzzy content-based group recommender system with dynamic selection of the aggregation functions," *International Journal of Approximate Reasoning*, vol. 150, pp. 273–296, 2022.
76 Huitzil, I., Alegre, F., and Bobillo, F., "Gimmehop: A recommender system for mobile devices using ontology reasoners and fuzzy logic," *Fuzzy Sets and Systems*, vol. 401, pp. 55–77, 2020.
77 Abbasi-Moud, Z., Hosseinabadi, S., Kelarestaghi, M., and Eshghi, F., "Cafob: Context-aware fuzzy-ontology-based tourism recommendation system," *Expert Systems with Applications*, vol. 199, p. 116877, 2022.
78 Walek, B. and Fajmon, P., "A hybrid recommender system for an online store using a fuzzy expert system," *Expert Systems with Applications*, vol. 212, p. 118565, 2023.

The Roles of Fuzzy Systems in Biomedical Engineering

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***Abstract*—Fuzzy set theory has been introduced for 60 years. Since then, there are many theories developed based on the main theory. These theories have been used in several sectors including in biomedical engineering applications. In this paper, we show some of algorithms developed based on fuzzy set theory. Also, some of biomedical engineering systems based on the existing fuzzy algorithms are shown.**

Keywords—Type-1 fuzzy logic, Type-2 fuzzy logic, Fuzzy patch-based segmentation, Neuro-fuzzy, Fuzzy co-occurrence matrix, Fuzzy vector pattern recognition

I. Introduction

Fuzzy set theory has been introduced by L. A. Zadeh [1]. Since then, there are many theories and applications involving the theory. Many research groups have used fuzzy set theory in many applications including in biomedical engineering research works. Our research group here in Chiang Mai University has also used and developed algorithms based on the fuzzy set theory and apply it in many applications including biomedical engineering applications. Hence, in this paper, we describe some of our works involving with fuzzy set theory either developing new algorithms or using existing fuzzy set algorithms in the biomedical engineering applications.

II. Developed Algorithms Based-On Fuzzy Set Theory

We have developed several algorithms based on fuzzy set theory and applied these algorithms in the biomedical engineering data sets. Examples of these algorithms are described as follows.

A. Fuzzy Patch-based Segmentation

The idea of fuzzy patch-based segmentation [2,3] is to overly segment an image using Fuzzy C-Means (FCM) clustering [4] with the number of clusters more than the number of classes. After that the required regions are derived by combining the patches in the oversegmented images. For example, if there are only two regions, i.e., nucleus and non-nucleus, the patch combining is achieved by considering the FCM centers. If the center of the patch is less than 60% of the mean of all centers, then the patch is labeled as nucleus. Otherwise, it is labeled as non-nucleus. This method was used in white blood segmentation [2,3], cervical cell classification [5], cardiac T2* relaxation time estimation [6], and spleen tissue segmentation [7]. Example of data set and result is shown in figure 1.

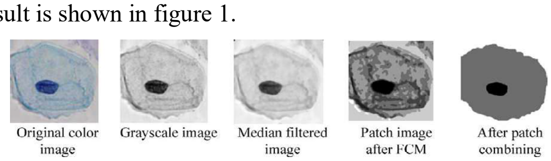


Fig. 1. Cervicle cell segmentation with fuzzy patch-based approach [5]

B. Neuro-Fuzzy Algorithm

There are several neuro-fuzzy algorithms developed by our research group such as gene selection and classification for diffuse large B-cell lymphomas microarrays [8] (model shown in figure 2). The model can achieve 100% classification rate on the validation sets of the 10-fold cross validation by using only 14 out of 7,070 features (genes) in the dataset. These 14 features including genes MDM4, STX16, NR1D2, DCLRE1A, PARK7, ATIC, HG4263-HT4533_at, CSRP1, NASP, PGK1, HLA-DPB1_2, HLA-A_2, ITK_2, and PLGLB2, were automatically selected by the model. The method could also identify the informative linguistic features for each class. For the DLBCL class, the first 5 most informative linguistic features were “HLA-A_2 is Small,” “NASP is Large,” “MDM4 is Small,” “ATIC is Medium,” and “STX16 is Small,” respectively. For class 2, the first 5 most informative linguistic features were “HLA-A_2 is Large,” “ATIC is Small,” “STX16 is Large,” “MDM4 is Medium,” and “NASP is Small,” respectively.

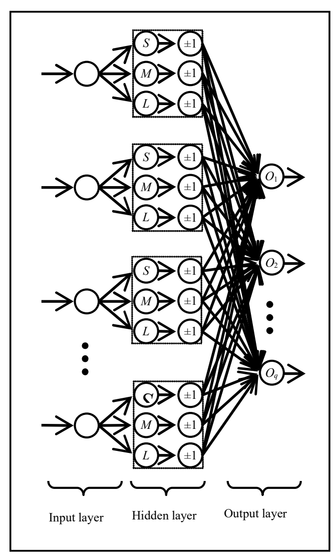


Fig. 2. A Neuro-fuzzy model in [8]

Another example of our neuro-fuzzy algorithms is the learning vector quantization inference classifier (LVQIC) [9] shown in figure 3. This model also provides 100% classification rate on the validation sets with 16 selected features.

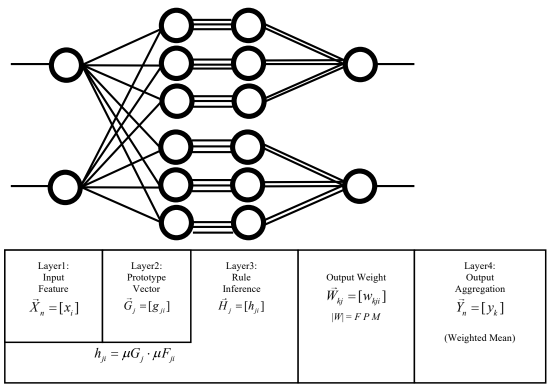


Fig. 3. The structure of LVQIC [z6]

C. *Fuzzy Co-occurrence Matrix (FCOM)*

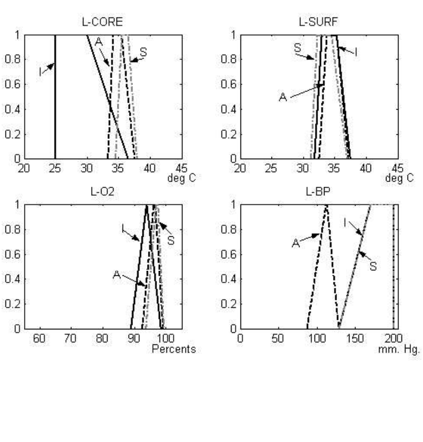
This method is one of the feature generation methods. The fuzzy C-means [4] is incorporated into the gray level co-occurrence matrix with 4 and 8 clusters. Then for 4 clusters, each FCOM [10] will have the size of 4×4, while that with 8 clusters will have the size of 8×8. Hence, there are 16 and 32 FCOM planes in total, respectively. Similar to those computed from the Gray Level Co-occurrence Matrix (GLCM), in each FCOM plane, we compute 14 features [11]. These features are used in LVQIC [12] mentioned in the previous section for abnormalities detection (architectural distortion (AD), asymmetry (ASYM), calcification (CALC), well-defined/circumscribed masses (CIRC), other ill-defined masses (MISC), and spiculated mass (SPIC)) in mammograms. Example of the blind test result of the model is shown in table 1.

TABLE I. THE BEST BLIND TEST RESULT FROM THE MODEL IN [12]

AD	SPIC	CALC	CIRC
100% with 0.03 FPR	100% with 0.04 FPR	100% with 0.06 FPR	100% with 0.02 FPR

D. *Fuzzy Vector Pattern Recognition*

Normally, data is in the form of a real number and it can also be represented by a vector of real numbers corresponding to an appropriate linear space. However, if there is an uncertainty in the data itself. This type of data can be produced by the imprecision of an agent who collected the data, or produced by nature, and can best be modeled by a fuzzy subset, the value of a linguistic variable. As a vector of fuzzy subsets in the Euclidean space, a linguistic vector can represent this uncertain data.



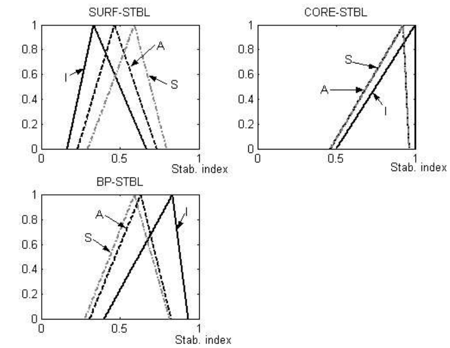


Fig. 4. Prototypes for I, A, and S Classes [13]

The fuzzy perceptron with pocket Algorithm [14] and fuzzy multilayer perceptron with cuckoo search [15] for the same data set were also developed. Figure 5 shows the fuzzy weight from the model in [14].

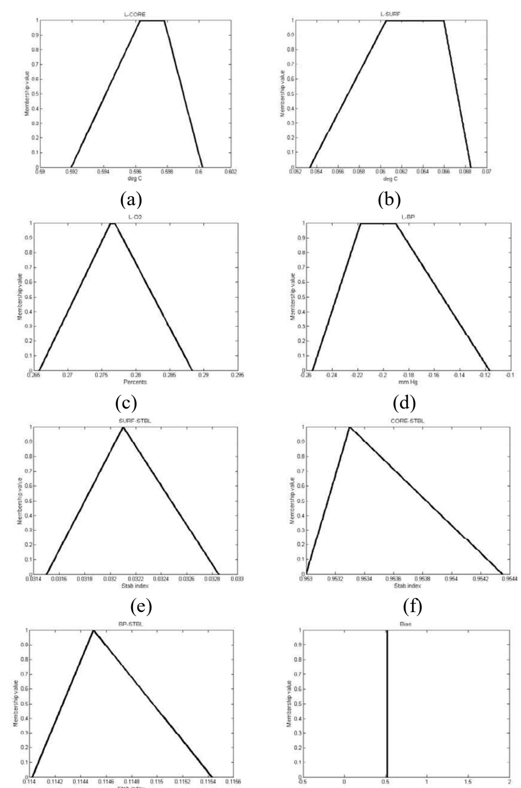


Fig. 5. Final weight fuzzy vectors [14] (a) L-CORE, (b) L-SURF, (c) L-O2, (d) L-BP, (e) SURF-STBL, (f) CORE-STBL, (g) BP-STBL, and (h) bias.

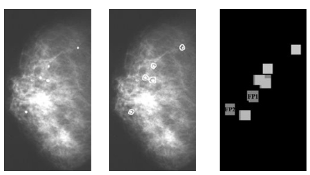
Example of this data in biomedical engineering is the postoperative patient data collected at the University of Kansas and the University of Missouri-Columbia. The linguistic hard C-means (LHCM) [13] was developed to group patients into Intensive Care Unit (I class), general hospital floor (A class) or go home (S class). Figure 4 shows the prototypes from the LHCM.

III. UTILIZATION OF THE EXISTING FUZZY SET ALGORITHMS

In this section, we summarize the utilization of existing fuzzy set algorithms, e.g., type-1 and type-2 fuzzy logic systems and fuzzy clustering algorithms in some of our biomedical engineering applications.

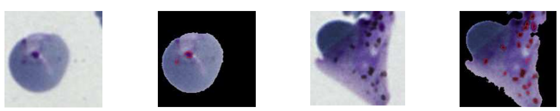
A. Type-1 and Type-2 Fuzzy Logic System

Type-1 fuzzy logic system, meaning the Mamdani system [16] in our case, has been used in many applications. We utilized this system in detecting microcalcification and mass in mammograms data set [17, 18]. In the same application, we also implemented the Mamdani fuzzy inference system with automatic membership function generation [19, 20] using the Possibilistic C-Means and compared the results with that using the Fuzzy C-Means applying to the same set of features [21]. Example of data set and results from this system is shown in figure 6. Other than detecting abnormalities in mammograms, we utilized the Mamdani system with automatically generated rules from the Wang-Mendel (WM) method [22, 23] and automatically generated membership function using the possibilistic fuzzy C-Means [23] in detection 4 infected classes (abnormal case) against non-infected class (normal case) [24] of an infected cell from *Plasmodium vivax* which is one of the *Plasmodium* species responsible for malaria in humans. Example of data set and results in this case is shown in figure 7.



(a) (b) (c)

Fig. 6. Example of mamogram data set (a) original, (b) expert's opinion, and (c) detection reuslts from the system [21].



original image (infected cell) | segmented cell (classified as infected cell) | original image (infected cell) | segmented cell (classified as infected cell)

(a)

original image (non-infected cell) | segmented cell (classified as non-infected cell)

(b)

Fig. 7. Examples of correct clssifications [24] (a) infected cells and (b) non-infected cell.

We also used Type-2 fuzzy logic system [25] in microcalcification detection system [26]. Moreover, we also implemented the same system with automatically generated membership functions [19, 20] using the Possibilistic C-Means and again compared the results with that using the Fuzzy C-Means [4] applying to the same set of features [27]. The results from the type-2 fuzzy logic system show that it is better than the system with type-1 fuzzy logic system as shown in figure 8. Another application of the type-2 fuzzy logic system is the lung nodule detection [28] which is a crucial task in lung cancer examination. Example of the data set and result is shown in figure 9.

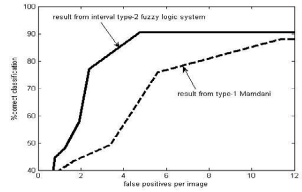


Fig. 8. ROC curve of microcalcification detection system [28] (dashed line show the result from type-1 Mamdani fuzzy inference system whereas solid line show the result from interval type-2 fuzzy logic system)

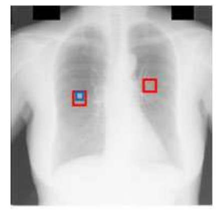


Fig. 9. Example of correct detection [28] (blue box is the ground truth, whereas the red box is the system output).

B. Fuzzy Clustering Algorithms

One of the health-related applications is a dental fluorosis classification system. Dental fluorosis occurs in many parts of the world because of highly exposure to high concentration of fluoride in the teeth development stage. To help the health policy makers developing the prevention and treatment plans, a manual or automatic image-based dental fluorosis classification system is needed. To build the system, we implemented multi-prototypes from Fuzzy C-means [19] in our system [29]. Also, in the same problem, we implemented [30] multi-prototype from the unsupervised possibilistic fuzzy clustering [31,32] via Cuckoo Search Algorithm [33,34]. We were able to classify the dental fluorosis into required stages. Example of data set and results [30] is shown in figure 10.

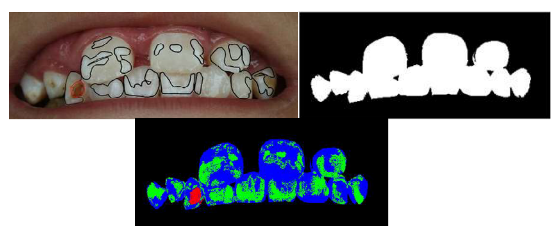


Fig. 10. Example of each fluorosis class from the training set and their segmented images [30]: Image F3_1 (Stage 3). The left image presents the expert's labels of opaque pixels and brown pixels, encircled in black and red, respectively. The middle image presents predicted binary tooth masks. The right image presents the predicted white-yellow, opaque, and brown pixels in blue, green, and red colors, respectively.

IV. Conclusion

To celebrate the 60th anniversary fuzzy set, in this paper, we describe some of our works involving with fuzzy set theory either developing new algorithms or using existing fuzzy set algorithms in biomedical engineering applications. In the future, there will certainly be more algorithms developed based on the fuzzy set theory. In addition, they will be used in many more applications in biomedical engineering, and in other fields.

ACKNOWLEDGMENT

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REFERENCES

[1] L.A. Zadeh, "Outline of new approach to the analysis of complex systems and decision processes," IEEE Trans. on Systems, Man and Cybernetics, 3(1), 1973.

[2] N. Theera-Umpon, "Patch-Based White Blood Cell Nucleus Segmentation Using Fuzzy Clustering," ECTI Transactions on Electrical Engineering, Electronics, and Communications Vol.3, No.1, pp.15–19, February 2005.

[3] N. Theera-Umpon, and S. Auephanwiriyakul, "Applying Fuzzy Clustering to Image Segmentation," Proceedings of the 2013 International Conference on the Frontiers of Information Technology, Application and Tools, 4 pg., Cheongju, Korea, September 2013.

[4] J.C. Bezdek, Pattern Recognition with Fuzzy Objective Function Algorithms; Plenum Press: New York, 1981.

[5] T. Chankong, N. Theera-Umpon, and S. Auephanwiriyakul, "Automatic Cervical Cell Segmentation and Classification in Pap Smears," Computer Methods and Programs in Biomedicine, Vol. 113, No. 2, pp. 539-556, February 2014.

[6] K. Wantanajittikul, N. Theera-Umpon, S. Saekho, S. Auephanwiriyakul, A. Phrommintikul, K. Leemasawat, "Automatic Cardiac T2* Relaxation Time Estimation from Magnetic Resonance Images Using Region Growing Method with Automatically Initialized Seed Points," Computer Methods and Programs in Biomedicine, Vol. 130, pp. 76-86, July 2016.

[7] P. Wuttisarnwattana and S. Auephanwiriyakul, "Spleen Tissue Segmentation Algorithm for Cryo Imaging Data", Journal of Digital Imaging, November 2022.

[8] N. Eiamkanitchat, N. Theera-Umpon, and S. Auephanwiriyakul, "On Feature Selection and Rule Extraction for High Dimensional Data: A Case of Diffuse Large B-cell Lymphomas Microarrays Classification," Mathematical Problems in Engineering, Vol. 2015, Article ID 275831, 12 pages, October 2015.

[9] C. Maisen, S. Auephanwiriyakul, and N. Theera-Umpon, "A Novel Learning Vector Quantization Inference Classifier", Lecture Notes in Computer Science, Springer-Verlag Berlin Heidelberg, (the 9th Asian Conference, ACIIDS 2017; Intelligent Information and Database Systems), pp. 528 – 544, 3 – 5 April 2017.

[10] Y. Munklang, S. Auephanwiriyakul, and N. Theera-Umpon, "Examination of Mammogram Image Classification using Fuzzy Co-occurrence Matrix," International Journal of Tomography and Simulation, Vol. 28, No. 3, pp. 96-103, 1 September 2015.

[11] R. M. Haralick, K. Shanmugam, and I. Dinstein, "Textural Features for Image Classification," IEEE Transactions on Systems, Man, and Cybernetics, SMC-3, 610-621, 1973.

[12] C. Maisen, S. Auephanwiriyakul, and N. Theera-Umpon, "Learning Vector Quantization Inference Classifier in Breast Abnormality Classification," Journal of Intelligent and Fuzzy Systems, vol. 35, no. 6, pp. 6101-6116, December 2018.

[13] S. Auephanwiriyakul, and N. Theera-Umpon, "Comparison of Linguistic and Regular Hard C-Means in Postoperative Patient Data," International Journal of Advanced Computational Intelligence and Intelligent Informatics, Vol.8, No.6, pp. 599–605, November 2004.

[14] S. Auephanwiriyakul, "A Preliminary Investigation of a Linguistic Perceptron", Lecture Notes in Computer Science/Lecture Notes in Artificial Intelligence (LNCS/LNAI), Vol. 3809, Springer, December 2005, pp. 1180-1186.

[15] S. Phitakwinai, S. Auephanwiriyakul, and N. Theera-Umpon. "Fuzzy Multilayer Perceptron with Cuckoo Search", JP Journal of Heat and Mass Transfer, Special Volume, Issue II, Advances in Mechanical System and ICI-convergence, July 2018, pp. 257 – 275.

[16] G. J. Klir and B. Yuan, Fuzzy Sets and Fuzzy Logic: Theory and Applications, Prentice Hall, Upper Saddle River, NJ., 1995.

[17] S. Auephanwiriyakul, S. Attrapadung, S. Thovutikul, and N. Theera-Umpon, "Calcification Detection in Mammograms using Fuzzy Inference System", EECON2004, Vol. 2, Khon Kaen, Thailand, November 2004, pp 165-168.

[18] S. Auephanwiriyakul, S. Attrapadung, S. Thovutikul, and N. Theera-Umpon, "Breast Abnormality Detection in Mammograms Using Fuzzy Inference System," 2005 IEEE International Conference on Fuzzy Systems, Reno, Nevada, U.S.A., May 2005, pp.155-160.

[19] R. Krishnapuram, and J. Keller, "A possibilistic approach to clustering," IEEE Trans. Fuzzy Systems, Vol. 1, pp. 98–110, Feb 1993.

[20] R. Krishnapuram, and J. Keller, "Fuzzy and Possibilistic Clustering Methods for Computer Vision," Neural and Fuzzy Systems, SPIE Press, pp. 133-159, 1994.

[21] S. Chumklin, S. Auephanwiriyakul, and N. Theera-Umpon, "Microcalcification Detection in Mammograms Using Fuzzy Inference System with Automatic Membership Function Generation", The 4th International Symposium on Biomedical Engineering (IEEE ISBME 2009), Bangkok, Thailand, December 2009.

[22] L. X. Wang and J. M. Mendel, "Generating Fuzzy Rules by Learning from Examples," IEEE Trans. Syst. Man Cybern., vol. 22, no. 6, pp. 1414–1427, 1992.

[23] D. Alvarez-Estevez and V. Moret-Bonillo, "Revisiting the Wang–Mendel algorithm for fuzzy classification," Expert Syst., vol. 35, no. 4, 2018.

[24] S. Turakit, S. Auephanwiriyakul and N. Theera-Umpon, "Plasmodium Vivax in Thin Blood Smear Classification Using Fuzzy Inference System With Automatic Membership Functions and Rules Generation", The 21st International Conference on Electrical Engineering/Electronics, Computer, Telecommunications and Information Technology (ECTI-CON) 2024, Khon Kaen, Thailand, May 27 – 30, 2024.

[25] J. M. Mendel, Uncertain Rule-Based Fuzzy Logic System: Introduction and New Directions, Prentice Hall, Upper Saddle River, NJ, 2001.

[26] S. Thovutikul, S. Auephanwiriyakul, and N. Theera-Umpon, "Microcalcification Detection in Mammograms Using Interval Type-2 Fuzzy Logic System", 2007 IEEE International Conference on Fuzzy Systems, London, UK, July, 2007.

[27] S. Chumklin, S. Auephanwiriyakul, and N. Theera-Umpon, "Microcalcification Detection in Mammograms Using Interval Type-2 Fuzzy logic System with Automatic Membership Function Generation," Proceedings of the 2010 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2010) in part of the 2010 IEEE World Congress on Computational Intelligence (WCCI 2010), Barcelona, Spain, July 2010.

[28] K. Suttitanawat, A. Uppanun, S. Auephanwiriyakul, N. Theera-Umpon, and P. Wuttisarnwattana, "Lung Nodule Detection from Chest X-Ray Images Using Interval Type-2 Fuzzy Logic System, Proceedings of the 2018 IEEE International Conference on Control System, Computing and Engineering, Penang, Malaysia, November 2018.

[29] U. Yeesarapat, S. Auephanwiriyakul, N. Theera-Umpon, and C. Kongpun, "Dental Fluorosis Classification Using Multi-prototypes from Fuzzy C-Means Clustering", 2014 Conference on Computational Intelligence in Bioinformatics and Computational Biology, Hawaii, USA., May 21 – 24, 2014.

[30] R. Wongkhuenkaew, S. Auephanwiriyakul, N. Theera-Umpon, K. Teeyapan, and U. Yeesarapat, "Fuzzy K-Nearest Neighbor Based Dental Fluorosis Classification Using Multi-Prototype Unsupervised Possibilistic Fuzzy Clustering via Cuckoo Search Algorithm", International Journal of Environmental Research and Public Health, 20(4), 3394, 2023.

[31] X. Wu, B. Wu, J. Sun, H. Fu, "Unsupervised Possibilistic Fuzzy Clustering", Journal of Information and Computational Science 2010, 7, 1075–1080.

[32] Z. Cebeci, A. T. Kavlak, F. Yildiz, "Validation of Fuzzy and Possibilistic Clustering Results" In Proceedings of the 2017 International Artificial Intelligence and Data Processing Symposium (IDAP); September 2017; pp. 1–7.

[33] X.-S. Yang, S. Deb, "Cuckoo Search via Lévy Flights" In Proceedings of the 2009 World Congress on Nature Biologically Inspired Computing (NaBIC); December 2009; pp. 210–214.

[34] X.-S. Yang, S. Deb, "Engineering Optimisation by Cuckoo Search" International Journal of Mathematical Modelling and Numerical Optimisation 2010, 1, 330–343, doi:10.1504/IJMMNO.2010.03543.

Trends on information fusion under uncertainty

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***Abstract*—In this note we recall some important concepts on aggregation techniques that have changed the traditional methodology for information fusion under uncertainty, namely, the pre-aggregation functions that allow directional monotonicity, (a, b)-fusion functions, which allow to define known aggregation functions in any real interval without losing their constituent features and properties, and width-limited interval-valued aggregation functions that allows to control the information quality of the results in interval-valued problems.**

***Index Terms*—Aggregation functions, pre-aggregation functions, Width-limited interval-valued aggregation functions, (a,b)-aggregation functions**

I. Introduction

Information fusion is featured in many different fuzzy models, where, numerous membership values have to be combined into a single representative one, according to some criteria. This process is usually carried out by aggregation functions, which are increasing operators with some boundary conditions.

The study of aggregation functions has been the focus of many researchers of the fuzzy community since the origin of the concept. In fact, data aggregation is involved in several applications, such as classification, image processing, multi-criteria decision making, analysis of social networks, brain-computer interface, control, convolutional neuro networks and adaptive neuro fuzzy networks.

Such models rely on well-developed aggregation mechanisms, and recently some of them have faced new proposals. For that reason, by introducing new aggregation functions, many practical applications benefited from the definition of new aggregation operators, motivating the theoretical development of such functions.

Here, we will highlight 3 works (namely [1]–[3]), from the last decade that we think deserve the attention of the fuzzy community and that could inspire new developments in the field of aggregation-like functions.

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II. Pre-aggregation functions and weaker forms of monotonicity

Traditionally, aggregation functions have the monotonicity property. However, important operators such as the mode are not increasing, but can still be applied to fuse information. Inspired by this situation, researchers started to study alternative forms of monotonicity, such as weak monotonicity and directional monotonicity, which allow monotonicity to be fulfilled along (some) fixed ray.

With this in mind, Lucca et al. [1] have introduced the concept of pre-aggregation functions. These functions respect the boundary condition as any aggregation function; however, they are directional increasing. This novel concept allowed the definition of generalized functions based on fuzzy integrals that have been successfully applied, primarily in classification problems [4]–[7], but also in other applications such as image processing [8], [9], multi-criteria decision making [10], [11] and brain-computer interface [12], [13].

It should be noted that, to this day, a state-of-the-art fuzzy rule-based classification system relies on a pre-aggregation function in a crucial aggregation step of its inference method [14]. In addition, the state-of-the-art method for the edge detection problem adopts pre-aggregation functions given by generalized fuzzy integrals (based on sliding window adaptive fuzzy measures), which are used to merge extracted features [15].

III. Width-limited interval-valued aggregation functions

The concept of aggregation function has been extended to consider other type of inputs, when dealing with different types of fuzzy modeling. For example, when considering interval-valued fuzzy sets, interval-valued aggregation functions have to be applied to enact the information fusion of the interval membership degrees. In this context, the width of the operated intervals can carry the quality of the information of the process.

In Asmus et al. [2], a theoretical framework was introduced to define interval-valued fusion functions with information quality control, in a way that the defined interval function is based on a core fusion function (defined in $[0, 1]$) and the widths of the operated intervals are limited by a width-limiting function. This framework enables the definition of a flexible interval aggregation operator with parameters that can be adjusted according to the application at hand. This methodology has been applied in interval-valued fuzzy problems, for example, for attribute reduction in incomplete interval-valued information systems [16].

Additionally, the application of interval overlap operators with controlled widths improved the accuracy of the IVTURS method, which is now the state-of-the-art for interval-valued fuzzy rule-based classification systems [2].

IV. (a, b)-AGGREGATION FUNCTIONS

It is known that aggregation functions can be defined in any real interval [a,b] by adjusting the boundary conditions and that some relevant practical applications have to aggregate inputs that are not from the unit interval, such as in convolutional networks. However, since most of the developments in the field of aggregation functions revolve around some type of fuzzy modeling, it is natural that most of the functions used are defined in the unit interval $[0, 1]$.

In order to appropriately transpose some fuzzy operators to be applied in other domains, beyond the unit interval, Asmus et al. [3] have introduced a general framework to characterize classes of fusion functions with floating domains, called (a, b)-fusion functions, defined in any closed real interval $[a, b]$, based on classes of core fusion functions defined on $[0, 1]$. The fundamental aspect of this framework is that the properties of a core aggregation function are preserved in the context of the analogous (a, b)-aggregation function.

The concepts of this work enabled the definition, for example, of n-dimensional (a, b)-grouping functions, which have provided excellent results when applied as the pooling operator in convolutional neural networks [17]–[19].

V. FINAL THOUGHTS

With Artificial Intelligence at the forefront of recent technological advancements, the field of aggregation functions is as relevant as ever. Concepts from the key highlighted papers, namely [1]–[3], are currently applied to further develop the aggregation step of many computational methods. We chose them since they are not bound to a specific problem or application; on the contrary, they introduce constructive approaches to develop new, flexible, aggregation operators.

We hope that this brief exposition can introduce what we consider to be some of the most relevant developed concepts from the last decade in the field of aggregation functions to some colleagues of the fuzzy community, maybe inspiring new approaches and studies in this field.

REFERENCES

[1] G. Lucca, J. Sanz, G. Pereira Dimuro, B. Bedregal, R. Mesiar, A. Kolesárová, and H. Bustince Sola, "Pre-aggregation functions: construction and an application," *IEEE Transactions on Fuzzy Systems*, vol. 24, no. 2, pp. 260–272, April 2016.

[2] T. da Cruz Asmus, J. A. Sanz, G. P. Dimuro, J. Fernandez, R. Mesiar, and H. Bustince, "A methodology for controlling the information quality in interval-valued fusion processes: Theory and application," *Knowledge-Based Systems*, vol. 258, p. 109963, 2022.

[3] T. da Cruz Asmus, G. P. Dimuro, B. Bedregal, J. A. Sanz, J. Fernandez, I. Rodriguez-Martinez, R. Mesiar, and H. Bustince, "A constructive framework to define fusion functions with floating domains in arbitrary closed real intervals," *Information Sciences*, vol. 610, pp. 800–829, 2022.

[4] J. Wieczynski, G. Lucca, G. P. Dimuro, E. N. Borges, J. A. Sanz, T. d. C. Asmus, J. Fernández, and H. Bustince, "dc_F-integrals: Generalizing c_F-integrals by means of restricted dissimilarity functions," *IEEE Transactions on Fuzzy Systems*, vol. 31, no. 1, pp. 160–173, 2023.

[5] J. Wieczynski, G. Lucca, E. Borges, L. Emmendorfer, M. Ferrero-Jaurrieta, G. Dimuro, and H. Bustince, "Applying d-xchoquet integrals in classification problems," in *2022 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*, 2022, pp. 1–7, (Best Paper Award).

[6] T. Batista, B. Bedregal, and R. Moraes, "Constructing multi-layer classifier ensembles using the choquet integral based on overlap and quasi-overlap functions," *Neurocomputing*, vol. 500, pp. 413–421, 2022.

[7] J. Fumanal-Idocin, O. Cordón, and H. Bustince, "The krypteia ensemble: Designing classifier ensembles using an ancient spartan military tradition," *Information Fusion*, vol. 90, pp. 283–297, 2023.

[8] C. Marco-Detchart, G. Lucca, C. Lopez-Molina, L. De Miguel, G. Pereira Dimuro, and H. Bustince, "Neuro-inspired edge feature fusion using Choquet integrals," *Information Sciences*, vol. 581, pp. 740–754, 2021.

[9] S. R. Hait, R. Mesiar, P. Gupta, D. Guha, and D. Chakraborty, "The bonferroni mean-type pre-aggregation operators construction and generalization: Application to edge detection," *Information Fusion*, vol. 80, pp. 226–240, 2022.

[10] J. Wieczynski, J. Fumanal-Idocin, G. Lucca, E. N. Borges, T. d. C. Asmus, L. R. Emmendorfer, H. Bustince, and G. P. Dimuro, "d-xc integrals: On the generalization of the expanded form of the choquet integral by restricted dissimilarity functions and their applications," *IEEE Transactions on Fuzzy Systems*, vol. 30, no. 12, pp. 5376–5389, 2022.

[11] C. Durán, A. K. Yazdi, I. Derpich, and Y. Tan, "Leveraging blockchain for maritime port supply chain management through multicriteria decision making," *Mathematics*, vol. 12, no. 10, 2024.

[12] L. Ko, Y. Lu, H. Bustince, Y. Chang, Y. Chang, J. Fernández, Y. Wang, J. A. Sanz, G. P. Dimuro, and C. Lin, "Multimodal fuzzy fusion for enhancing the motor-imagery-based brain computer interface," *IEEE Comput. Intell. Mag.*, vol. 14, no. 1, pp. 96–106, 2019. [Online]. Available: https://doi.org/10.1109/MCI.2018.2881647

[13] J. Fumanal-Idocin, Y.-K. Wang, C.-T. Lin, J. Fernández, J. A. Sanz, and H. Bustince, "Motor-imagery-based brain–computer interface using signal derivation and aggregation functions," *IEEE Transactions on Cybernetics*, vol. 52, no. 8, pp. 7944–7955, 2022.

[14] G. Lucca, G. P. Dimuro, J. Fernandez, H. Bustince, B. Bedregal, and J. A. Sanz, "Improving the performance of fuzzy rule-based classification systems based on a nonaveraging generalization of CC-integrals named $C_{F_1F_2}$-integrals," *IEEE Transactions on Fuzzy Systems*, vol. 27, no. 1, pp. 124–134, Jan 2019.

[15] C. Marco-Detchart, G. Lucca, M. A. Santos Silva, J. A. Rincon, V. Julian, and G. Dimuro, "Sliding window based adaptative fuzzy measure for edge detection," *Expert Systems*, p. e13730, 2024.

[16] J. Dai, Z. Wang, and W. Huang, "Interval-valued fuzzy discernibility pair approach for attribute reduction in incomplete interval-valued information systems," *Information Sciences*, vol. 642, p. 119215, 2023.

[17] I. Rodriguez-Martinez, T. da Cruz Asmus, G. P. Dimuro, F. Herrera, Z. Takáč, and H. Bustince, "Generalizing max pooling via (a,b)-grouping functions for convolutional neural networks," *Information Fusion*, vol. 99, p. 101893, 2023.

[18] I. Rodriguez-Martinez, P. Ursua-Medrano, J. Fernandez, Z. Takáč, and H. Bustince, "A study on the suitability of different pooling operators for convolutional neural networks in the prediction of covid-19 through chest x-ray image analysis," *Expert Systems with Applications*, vol. 235, p. 121162, 2024.

[19] M. Li, X. Zhang, H. Jiang, and J. Liu, "Some novel fuzzy logic operators with applications in fuzzy neural networks," *Information Sciences*, vol. 702, p. 121897, 2025.

On behalf of fuzzy approximate dependencies and related extensions

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***Abstract*—This paper is devoted to the recognition of the determinant work of pioneers in fuzzy logic in the field of knowledge discovery in databases. When knowledge comes from real-world data, it can be affected by noise, imprecision, and uncertainty, and hence, existing methods need to be properly adapted in order to manage this issue. Well-known techniques such as association rule mining have been extended to the fuzzy case, allowing not only to discover fuzzy association rules, but also different types of knowledge, as it is the case of fuzzy approximate dependencies and fuzzy gradual dependencies. Fuzzy sets theory has also opened the door to different approaches for uncertainty and imprecision management in data mining, such as the ones based on representation by restriction levels. Finally, the advantages of a novel framework for incremental maintenance of fuzzy rules are discussed.**

***Index Terms*—fuzzy data mining, fuzzy rules, fuzzy dependencies**

I. Introduction

Within the vast field of knowledge discovery techniques, association rules (ARs) have been consolidated as one of the most popular and well-known. ARs have been widely used for discovering hidden patterns and relationships among itemsets in large volumes of data. ARs are defined over transactions of items, and were originally conceived [1] to analyze market baskets (e.g. *"customers who buy diapers also buy beer"*), and their scope has expanded significantly. In particular, when applied to real-world data, which are frequently noisy, incomplete, or imprecise, classical ARs can be not enough, often resulting in the loss of valuable information or the generation of artificially, ill-defined, rules.

Fuzzy set theory, as proposed by L.A. Zadeh [2], introduced a new paradigm able to manage uncertainty and imprecision in data by allowing elements to belong to a set with a degree of membership between 0 and 1. This flexibility proved fundamental for knowledge discovery in databases [3], where attributes can be inherently ambiguous (e.g., *"high temperature"*, *"average salary"*) or where the goal is to capture relationships that are not strictly binary.

Returning to association rules, as powerful as they are in terms of describing information, further research has allowed them to be applied in the discovery of different types of knowledge. One particular case is that of approximate dependencies (ADs), which can be viewed as *"functional dependencies with exceptions"*. In the field of databases, a functional dependency (FD) between two or more attributes occurs when the values of some of these attributes determine the values of other attributes. However, in real-world databases, patterns are rarely perfect. There may be exceptions, noise, or inherent variations that make a relationship "mostly" true, but not absolutely. Hence, ADs allow to flexibilize this condition. In [4], it is stated how, by transforming a relational table into a set of transactions, it is possible to obtain approximate dependencies in terms of association rules.

Moreover, when dealing with imprecise data, classical methods for AR extraction can be extended to the fuzzy case. In the same way, the methodology mentioned before can be extended to obtain other types of fuzzy rules in terms of fuzzy association rules. In this work, our intention is to acknowledge Zadeh's seminal work, the following research, and its role in the definition and development of new methods and techniques for knowledge discovery in fuzzy databases, by extending the definition of association rules.

The paper is structured as follows. First, we recall the classical definition for ARs and the definition of ADs in terms of the former. Then, we recall their extensions to the fuzzy case, as well as discuss some alternate and open research lines.

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II. From crisp association rules to approximate dependencies

In this section, we recall the main definitions regarding association rules, as a data mining tool to extract hidden relations in transactions of items.

A. Fundamentals of association rules

Being I a finite set of items and T a finite set of transactions with items in I, an association rule, $A \Rightarrow B$, where $A, B \subseteq I$, $A, B \neq \emptyset$, and $A \cap B = \emptyset$, can be interpreted as *"every transaction in T that contains A also contains B"*. Since this statement may not hold for every transaction in T, measures are often defined to establish the interest, based on the support of the rule (fraction of transactions that contain $A \cup B$) and the accuracy or confidence of the rule (fraction of transactions that contain $A \cup B$ in transactions that contain A). However, different authors have pointed out some drawbacks on confidence as a misleading metric, and alternative metrics such as the certainty factor [5] have been proposed in the literature [6], [7].

B. Approximate dependencies as association rules

Let R be a set of attributes, and r be a relation with attributes in R. Given $X, Y \subset R$ with $X \cap Y = \emptyset$, a functional dependency $X \rightarrow Y$ holds in R iff for every instance r of R,

$$\forall t, s \in r, \, if \, t[X] = s[X] \, then \, t[Y] = s[Y]. \tag{1}$$

Looking for functional dependencies in relational databases has been an object of interest in the field of data mining, since they can be very informative about the inner structure of data. However, it is difficult to find perfect functional dependencies, since one single exception to 1 causes the dependency to not hold. Approximate dependencies allow to flexibilize this restriction, while still revealing us interesting regularities in data.

The definition of approximate dependencies is then a matter of how to consider these exceptions and how to measure their precision (that is, the proportion of tuples in a relation where the dependency holds), as seen in [8]. In the present case, following the proposal in [4], it is seen how approximate dependencies can be obtained in terms of association rules. This approach is also interesting because it allows us to apply the same metrics (support, confidence, and certainty factor) to measure the interest and accuracy of an AD.

III. From crisp (classical) to fuzzy rules

The transition from the classical to the fuzzy domain was a natural and necessary step to address the reality of databases with quantitative and qualitative attributes that cannot be adequately represented by discrete categories or exact values.

A. Fuzzy association rules

Fuzzy association rules (FARs) extend the classical concept, and different approaches can be found in the literature (see [9], [10] for a review). In [9], FARs are defined and assessed by defining transactions as fuzzy subsets of I, and considering membership degrees between 0 and 1 of items in a fuzzy transaction. In a similar way to the definition offered in subsection II-A, a fuzzy association rule can be seen as an implication of the form $A \Rightarrow B$, where $A, B \subseteq I$, $A, B \neq \emptyset$, and $A \cap B = \emptyset$. In this case, a semantic approach based on the evaluation of quantified sentences, as described in [11], can be used to define a set of fuzzy metrics that yield the ordinary measures for support, confidence and certainty factor in the crisp case. Hence, this approach offers a consistent fuzzy generalization of crisp association rules.

B. Fuzzy approximate dependencies

Following an analogous methodology as the one described in II-B, fuzzy approximate dependencies (FADs) are proposed in [12], and defined in terms of fuzzy association rules. In addition to allowing exceptions, several elements of the definition of functional dependencies (equation 1) are relaxed. In particular, membership degrees associated to pairs $< attribute, value >$ in a fuzzy relation are considered, as well as fuzzy similarity relations that smooth the equality of the rule.

One of the advantages of this approach is that of measuring the interest and accuracy of FADs in terms of FARs, as an extension to the fuzzy case of the method mentioned in section II-B, by means of the fuzzy measures mentiond in section III-A.

In [13], [14], FARs and FADs are applied to obtain correspondences between attributes in a fuzzy context, particularly an agricultural environment, as an extension of the problem of correspondence analysis to the fuzzy domain.

C. Additional fuzzy rules: gradual dependencies

Gradual dependencies can be seen as a special type of rules that represent a trend, a casual relation among the variation in the degree of fulfillment of imprecise properties by different objects [15]. Consider for instance a database containing data about vehicles. An example of gradual dependence is *"the higher the weight, the lower the speed"*, meaning that as the weight of a truck increases, its speed tends to decrease. In general, we can consider all types of gradual dependencies derived from the form *"the more/less X is A, the more/less Y is B"*, where X, Y are attributes that can be modeled by fuzzy properties A, B. Formally, a gradual dependence $(*_1, X, A) \rightarrow (*_2, Y, B)$ with $*_1, *_2 \in <, >$ is defined by equation 2.

$$\forall t, s \in r, \, if \, A(t[X]) *_1 A(s[X]) \, then \, B(t[Y]) *_2 B(s[Y]). \tag{2}$$

Based on suitable definitions of the abstract notions of item and transaction, the proposal in [16] allows to obtain fuzzy gradual dependencies as fuzzy association rules, and the measures of support, confidence and certainty factor are employed in order to assess such dependencies.

IV. Rule mining via Restriction Levels

Research into fuzzy association rules and their extensions continues to evolve, leading to approaches that address specific challenges or explore new perspectives in knowledge representation.

As discussed before, fuzzy sets theory offers a way to represent and manage concepts and objects affected by vagueness and imprecision. In the last years, an alternative theory, called representation by levels (RL) has been proposed (see [17], [18]) for the same purpose, covering some important capabilities that cannot be provided by fuzzy sets theory.

In short, instead of considering membership degrees, RLs assign objects to levels in an ordered set, where those levels represent degrees of relaxation of the criteria employed when defining properties or features.

As described in [19], [20], representation by levels offers an intuitive and straightforward way to obtain fuzzy association rules as well as other types of rules, by extracting association rules on each restriction level. The proposal ensures that crisp operations and measures can be easily extended to RLs, while maintaining all the properties of the crisp case.

This is particularly interesting due to different reasons:

- RLs allows to extract approximate dependencies at a certain restriction level. Users can be interesting in crisp approximate dependencies only: easier to understand, dependencies to be found only at a given level …
- Due to the nature of data itself, and derived from the RL approach, some attributes values may be present only at certain levels, and so may happen with the related dependencies. RLs should help in the discovery of this potentially useful knowledge.
- Obtained crisp approximate dependencies, extracted level by level, can be later aggregated into fuzzy approximate dependencies, without any loss of generality, thus allowing users to decide which ones can be more interesting.

When considering RLs, it must be remarked that in no way it is meant that conventional fuzzy data mining is not useful or must be replaced by using RLs. In [20] it is shown that RLs are a suitable alternative to fuzzy sets for representing fuzziness, as they offer very important properties and possibilities that fuzzy sets cannot, providing solutions when such properties are required. They must be seen as an alternative to consider, in the same way that there are different alternatives for fuzzy operators in fuzzy set theory when modeling a fuzzy system, and no one can be said to be better than another in general.

V. Incremental fuzzy data mining

One of the biggest challenges in data mining in dynamic environments is the efficient updating of extracted knowledge when the underlying database changes. Real-world databases, e.g., stream data, sensor network data, network traffic data, etc., constantly grow and are modified with new transactions, deletions, or updates. Regular knowledge discovery techniques can lead us to inexact or eventually obsolete rules. And recalculating all derived rules from scratch every time the database is modified is computationally expensive and often unfeasible for large data volumes.

This is where incremental data mining becomes an imperative need. This paradigm seeks to develop techniques that can efficiently update existing rules by incorporating changes in the database without the need to reprocess the entire dataset. The key is to reuse previously mined knowledge and only process the new or modified portions of the database.

In [21], a novel framework for maintenance of discovered rules based on incremental data mining is proposed, in order to address these issues. The advantage of this proposal lies in the fact that, through materialized views, it keeps the metrics of the extracted rules up to date, allowing for real-time detection of whether any of these rules are invalidated or become obsolete with the emergence of new evidence in data. Two approaches are described, an immediate incremental maintenance method, which updates the rule database every time the main database is updated, and a second, deferred incremental maintenance method, which allows to schedule the update in a more efficient way.

Furthermore, the representation of knowledge and its metrics is scalable enough to be extended to new types of knowledge and new metrics. In particular, the extension to the fuzzy case, for the maintenance of fuzzy association rules and fuzzy approximate dependencies, is discussed in [22].

VI. Concluding remarks and future perspectives

Without a doubt, L.A. Zadeh's fuzzy sets theory has proven to be a relevant milestone for the improvement of artificial intelligence, specially in the field of knowledge discovery in databases, reflecting the complexity and imprecision in real-world data. In this work, it is recalled how classical (crisp) knowledge as association rules and approximate dependencies can be extended to the fuzzy domain, allowing to handle the imprecision of quantitative and linguistic attributes. The exploration of fuzzy gradual dependencies allows extraction of trend patterns between attributes.

In addition, alternative approaches, as representation by levels, have been highlighted, as well as the crucial need for incremental fuzzy data mining for dynamic systems.

The relevance of these techniques extends to different practical applications, from decision-making in uncertain scenarios to customization in recommendation systems and diagnosis in expert systems. The field remains open for future challenges and opportunities, including:

- Development of more scalable techniques for fuzzy data mining in large volumes of data.
- Improvement of the interpretability of obtained rules, looking for more intuitive ways to present extracted knowledge to end users.
- Application of these techniques in emerging new domains, such as sensor networks and Internet of Things, personalized healthcare systems, and some other scenarios where uncertainty and imprecision are ubiquitous.

As a conclusion, fuzzy sets theory is not just a mathematical theory, but an invitation to embrace the complexity of real world, providing tools to extract meaningful and useful knowledge from uncertain and imprecise data, in the pursuit of more robust, adaptable and, particularly, more human intelligent systems.

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References

[1] R. Agrawal, T. Imieliński, and A. Swami, "Mining association rules between sets of items in large databases," in *Proceedings of the 1993 ACM SIGMOD international conference on Management of data*, 1993, pp. 207–216.

[2] L. A. Zadeh, "Fuzzy sets," *Information and control*, vol. 8, no. 3, pp. 338–353, 1965.

[3] W. Pedrycz, "Fuzzy set technology in knowledge discovery," *Fuzzy Sets and Systems*, vol. 98, no. 3, pp. 279–290, 1998.

[4] D. Sánchez, J. M. Serrano, I. Blanco, M. J. Martín-Bautista, and M.-A. Vila, "Using association rules to mine for strong approximate dependencies," *Data Mining and Knowledge Discovery*, vol. 16, pp. 313–348, 2008.

[5] E. H. Shortliffe and B. G. Buchanan, "A model of inexact reasoning in medicine," *Mathematical biosciences*, vol. 23, no. 3-4, pp. 351–379, 1975.

[6] F. Berzal, I. Blanco, D. Sánchez, and M.-A. Vila, "A new framework to assess association rules," in *Advances in Intelligent Data Analysis: 4th International Conference, IDA 2001 Cascais, Portugal, September 13–15, 2001 Proceedings 4*. Springer, 2001, pp. 95–104.

[7] ——, "Measuring the accuracy and interest of association rules: A new framework," *Intelligent Data Analysis*, vol. 6, no. 3, pp. 221–235, 2002.

[8] P. Bosc, L. Lietard, and O. Pivert, "Functional dependencies revisited under graduality and imprecision," in *1997 Annual Meeting of the North American Fuzzy Information Processing Society-NAFIPS (Cat. No. 97TH8297)*. IEEE, 1997, pp. 57–62.

[9] M. Delgado, N. Marín, D. Sánchez, and M.-A. Vila, "Fuzzy association rules: general model and applications," *IEEE transactions on Fuzzy Systems*, vol. 11, no. 2, pp. 214–225, 2003.

[10] D. Dubois, E. Hüllermeier, and H. Prade, "A systematic approach to the assessment of fuzzy association rules," *Data Mining and Knowledge Discovery*, vol. 13, pp. 167–192, 2006.

[11] M. Delgado, D. Sánchez, and M. A. Vila, "Fuzzy cardinality based evaluation of quantified sentences," *International Journal of Approximate Reasoning*, vol. 23, no. 1, pp. 23–66, 2000.

[12] F. Berzal, I. Blanco, D. Sanchez, J.-M. Serrano, and M.-A. Vila, "A definition for fuzzy approximate dependencies," *Fuzzy Sets and Systems*, vol. 149, no. 1, pp. 105–129, 2005.

[13] D. Sánchez, J. Serrano, M. Vila, V. Aranda, J. Calero, and G. Delgado, "Using data mining techniques to analyze correspondences between user and scientific knowledge in an agricultural environment," in *Enterprise information systems IV*, 2003, pp. 75–89.

[14] M. López-Díaz, M. Á. Gil, P. Grzegorzewski, O. Hryniewicz, J. Lawry, J. Calero, G. Delgado, J.-M. Serrano, D. Sánchez, and M.-A. Vila, "A proposal of fuzzy correspondence analysis based on flexible data mining techniques," in *Soft Methodology and Random Information Systems*. Springer, 2004, pp. 447–454.

[15] E. Hüllermeier, "Association rules for expressing gradual dependencies," in *Principles of Data Mining and Knowledge Discovery: 6th European Conference, PKDD 2002 Helsinki, Finland, August 19–23, 2002 Proceedings 6*. Springer, 2002, pp. 200–211.

[16] C. Molina, J. M. Serrano, D. Sánchez, and M.-A. Vila, "Mining gradual dependencies with variation strength," *Mathware & soft computing*, vol. 15, no. 1, pp. 75–93, 2008.

[17] D. Sánchez, M. Delgado, and M.-A. Vila, "A restriction level approach to the representation of imprecise properties," *IPMU 2008*, pp. 153–159, 2008.

[18] D. Sánchez, M. Delgado, M.-A. Vila, and J. Chamorro-Martínez, "On a non-nested level-based representation of fuzziness," *Fuzzy Sets and Systems*, vol. 192, pp. 159–175, 2012.

[19] M. Delgado, M. D. Ruiz, D. Sánchez, and J.-M. Serrano, "A formal model for mining fuzzy rules using the rl representation theory," *Information Sciences*, vol. 181, no. 23, pp. 5194–5213, 2011.

[20] C. Molina, M. D. Ruiz, and J. M. Serrano, "Representation by levels: An alternative to fuzzy sets for fuzzy data mining," *Fuzzy Sets and Systems*, vol. 401, pp. 113–132, 2020.

[21] A. Pérez-Alonso, I. J. B. Medina, L. M. González-González, and J. M. Serrano Chica, "Incremental maintenance of discovered association rules and approximate dependencies," *Intelligent Data Analysis*, vol. 21, no. 1, pp. 117–133, 2017.

[22] A. Pérez-Alonso, I. J. Blanco, J.-M. Serrano, and L. M. González-González, "Incremental maintenance of discovered fuzzy association rules," *Fuzzy Optimization and Decision Making*, vol. 20, no. 4, pp. 429–449, 2021.

Representations by Levels and Fuzzy Sets: a Symbiosis for Graduality Management

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***Abstract*—We discuss about the contribution of the theory of Representations by Levels (RLs), akin to Gradual Sets and the X-μ approach, for dealing with graduality. RLs offer features that Fuzzy Sets cannot like, among others: i) keeping the Boolean algebra structure of crisp sets, among other properties of the crisp case; ii) offering a direct and unique extension of operations from the crisp to the fuzzy case, and iii) allowing graduality to be represented for mathematical objects other than sets, like elements (particularly numbers). We advocate for a symbiosis with Fuzzy Sets in what we call RL-systems, in which Fuzzy Sets are employed as inputs and also as a summary that complement the outputs, whilst the RL theory is employed in the internal calculations. There are many research opportunities in both theoretical aspects and the development of practical applications using RLs and related theories.**

I. Introduction

The introduction of Fuzzy Sets by Zadeh sixty years ago served to widely disseminate in the scientific community the need to deal with graded concepts, that is, concepts whose fulfillment is a matter of degree, like *high temperature*, *around 3*, etc. At the same time, research developed in the fuzzy community since then has provided numerous mathematical tools for representing and managing graded concepts, with the goal of mimicking the way humans handle such concepts. Over the past sixty years, researchers in Fuzzy Set Theory (FST) have contributed a wealth of theoretical results and have applied them to solving real-world problems. And, no doubt, the fuzzy community will continue to do so.

It is well known that in FST, a graded concept A defined on a referential set X is represented by a fuzzy set, which is specified by a membership function that assigns degrees of membership to elements of X, typically taking the form:

$$\mu_A : X \to [0,1]. \tag{1}$$

In this contribution, our analysis focuses on the ideas behind a particular group of alternative representations of graded concepts that have also emerged in the fuzzy community: Gradual Sets [1], Representations by Levels (RLs) [2], [3], and the X-μ approach [4], [5], [6][1], with particular focus on RLs. The common thread running through these alternative proposals is the representation of graded concepts by using functions of the form:

$$\rho_A : (0,1] \to 2^X, \tag{2}$$

where 2^X represents the power set of X. That is, membership functions of the form shown in Eq. (1) associate degrees to elements of X, focusing on the latter individually. Following a different paradigm, Eq. (2) takes as starting point values in $(0,1]$ to which crisp subsets of X are associated as possible *crisp versions* of A.

But, what is the semantics of Eq. (2)? Following the theory of RLs, the values in $(0,1]$ represent how strict we are in determining the elements of X that satisfy A. The value 1 represent the maximum possible degree of restriction, that is, $\rho_A(1)$ is the crisp set of elements of X for which there is no doubt they satisfy A. On the other extreme, the value 0 has a semantics of *no restriction*, and hence is never considered in RLs, as it gives no information about A. The semantics of intermediate values is associated to the distance to the extremes, representing a higher level of restriction as they are closer to 1. The value 0.5, for instance, is halfway between being totally strict and being not strict at all.

Hence, RLs conceive the representation of graded concepts as a collection of crisp versions associated to different levels of restriction (or, conversely, tolerance) in the application of the criteria to determine which elements satisfy A.

For those familiar with FST, Eq. (2) will remind you of the α-cut representations of fuzzy sets. Indeed, given a fuzzy set μ_A, an associated RL can be obtained by defining $\rho_A(\alpha) = (\mu_A)_\alpha \ \forall \alpha \in (0,1]$, with $(\mu_A)_\alpha$ being the α-cut of μ_A. In this sense, Eq. (1) corresponds to the usually called *vertical view* of a fuzzy set μ_A, whilst the corresponding assignment of α-cuts to levels in $(0,1]$ can be associated to the so-called *horizontal view*, which is a useful but subordinate tool employed in FST for purposes like extending crisp operations to the gradual case.

However, the RL theory goes beyond fuzzy sets. Notably, in Eq. (2), the sets $\rho_A(\alpha)$ are not required to satisfy the nesting property of α-cuts of fuzzy sets, as we shall see. Therefore, the theory of RLs is not a tool within FST, but an alternative theory for representing and managing graduality outside of it, offering some possibilities that FST does not. However,

[1]A review of related works can be found in [1], [3]. More recently, a relation between Gradual Sets and Hesitant Fuzzy Sets have been studied in [7].

TABLE I

EIGHT RLS WITH $\Lambda = \{1, 0.9, 0.7, 0.5, 0.3, 0.1\}$ REPRESENTING GRADED CONCEPTS. COLUMNS FOR A, B, $A \wedge \neg A$, $A \vee \neg A$, AND $A \wedge B$ CAN BE SEEN AS NESTED α-CUTS OF FUZZY SETS (COLUMNS $A \wedge \neg A$ AND $A \vee \neg A$ YIELDING THE CRISP RESULTS EXPECTED UNDER A BOOLEAN STRUCTURE) [3].

α	$\rho_A(\alpha)$	$\rho_{\neg A}(\alpha)$	$\rho_B(\alpha)$	$\rho_{\neg B}(\alpha)$	$\rho_{A\wedge\neg B}(\alpha)$	$\rho_{A\wedge\neg A}(\alpha)$	$\rho_{A\vee\neg A}(\alpha)$	$\rho_{A\wedge B}(\alpha)$
1	$\{x_1\}$	$\{x_2, x_3, x_4, x_5\}$	$\emptyset$	X	$\{x_1\}$	$\emptyset$	X	$\emptyset$
0.9	$\{x_1\}$	$\{x_2, x_3, x_4, x_5\}$	$\{x_1\}$	$\{x_2, x_3, x_4, x_5\}$	$\emptyset$	$\emptyset$	X	$\{x_1\}$
0.7	$\{x_1, x_2\}$	$\{x_3, x_4, x_5\}$	$\{x_1\}$	$\{x_2, x_3, x_4, x_5\}$	$\{x_2\}$	$\emptyset$	X	$\{x_1\}$
0.5	$\{x_1, x_2\}$	$\{x_3, x_4, x_5\}$	$\{x_1, x_3\}$	$\{x_2, x_4, x_5\}$	$\{x_2\}$	$\emptyset$	X	$\{x_1\}$
0.3	$\{x_1, x_2, x_3\}$	$\{x_4, x_5\}$	$\{x_1, x_3, x_4\}$	$\{x_2, x_5\}$	$\{x_2\}$	$\emptyset$	X	$\{x_1, x_3\}$
0.1	$\{x_1, x_2, x_3, x_5\}$	$\{x_4\}$	$\{x_1, x_3, x_4\}$	$\{x_2, x_5\}$	$\{x_2, x_5\}$	$\emptyset$	X	$\{x_1, x_3\}$

TABLE II

α-CUT REPRESENTATIONS OF THE FUZZY SETS ASSOCIATED TO THE GRADED CONCEPTS A AND B IN TABLE I, AND OF THE FUZZY SETS OBTAINED BY OPERATING USING MINIMUM, MAXIMUM AND STANDARD NEGATION.

α	$(\mu_A)_\alpha$	$(\mu_{\neg A})_\alpha$	$(\mu_B)_\alpha$	$(\mu_{\neg B})_\alpha$	$(\mu_{A\wedge\neg B})_\alpha$	$(\mu_{A\wedge\neg A})_\alpha$	$(\mu_{A\vee\neg A})_\alpha$	$(\mu_{A\wedge B})_\alpha$
1	$\{x_1\}$	$\{x_4\}$	$\emptyset$	$\{x_2, x_5\}$	$\emptyset$	$\emptyset$	$\{x_1, x_4\}$	$\emptyset$
0.9	$\{x_1\}$	$\{x_4, x_5\}$	$\{x_1\}$	$\{x_2, x_5\}$	$\emptyset$	$\emptyset$	$\{x_1, x_4, x_5\}$	$\{x_1\}$
0.7	$\{x_1, x_2\}$	$\{x_3, x_4, x_5\}$	$\{x_1\}$	$\{x_2, x_4, x_5\}$	$\{x_2\}$	$\emptyset$	X	$\{x_1\}$
0.5	$\{x_1, x_2\}$	$\{x_3, x_4, x_5\}$	$\{x_1, x_3\}$	$\{x_2, x_3, x_4, x_5\}$	$\{x_2\}$	$\emptyset$	X	$\{x_1\}$
0.3	$\{x_1, x_2, x_3\}$	$\{x_2, x_3, x_4, x_5\}$	$\{x_1, x_3, x_4\}$	$\{x_2, x_3, x_4, x_5\}$	$\{x_2, x_3\}$	$\{x_2, x_3\}$	X	$\{x_1, x_3\}$
0.1	$\{x_1, x_2, x_3, x_5\}$	$\{x_2, x_3, x_4, x_5\}$	$\{x_1, x_3, x_4\}$	X	$\{x_1, x_2, x_3, x_5\}$	$\{x_2, x_3, x_5\}$	X	$\{x_1, x_3\}$

this does not mean that one theory is better than the other in general. And, as we pointed out in the title of our contribution, both can complement each other, creating a synergy to address graduality in solving problems where FST's limitations prevent its use.

In the following Sections we informally overview some aspects of interest of RLs.

II. GRADUALITY IN SETS

We have seen in the previous section how both FST and RLs represent graduality in sets, providing different representation of the collection of elements of X that satisfy a graded concept A. Whilst FST can only represent graduality of sets, RLs can be applied to other mathematical objects, as we shall see in Section III.

A. Representation

Gradual Sets and the X-μ approach consider in general an infinite amount of crisp representatives of a graded concept A (think for instance of the collection of α-cuts of a trapezoidal fuzzy set defined on the real line). On its turn, RLs consider only the case of a finite amount of them. This is reasonable from a practical perspective, since the amount of levels that humans and computers are able to manage is always finite. The crisp representatives are associated to a collection of levels $\Lambda = \{\alpha_1, \ldots, \alpha_m\}$ with $1 = \alpha_1 > \alpha_2 > \cdots > \alpha_m > \alpha_{m+1} = 0$. Then, assuming that $\rho_A(\alpha_i)$ is given for every $\alpha_i \in \Lambda$, the function ρ_A is defined for every $\alpha \in (0, 1]$ as follows:

$$\rho_A(\alpha) = \rho_A(\alpha_i) \text{ iff } \alpha_i \geq \alpha > \alpha_{i+1}. \tag{3}$$

Table I shows eight *RL-sets* (RLs of sets) representing the fulfilment of eight different graded concepts on a set $X = \{x_1, \ldots, x_5\}$. A set Λ with six levels has been employed though, for some of the RL-sets, it would have sufficed with less levels. For instance, the RL-set for concept A only requires to provide the sets for levels 1, 0.7, 0.3 and 0.1. Even more, $A \wedge \neg A$ and $A \vee \neg A$ only require level 1, as they are crisp sets.

In Table I we can see that not all RL-sets have their crisp representatives nested as the collection of α-cuts of a fuzzy set. Three of them are not nested, and hence they do not correspond to a kind of *horizontal view* of fuzzy sets.

B. Operations

An important point that can be also illustrated with Table I is that of operations with RLs. The philosophy of RLs is that operations between RLs are performed by applying the operations using the crisp representatives in each level independently. As an example, the complement of the RL-set for concept A, corresponding to the representation of $\neg A$, is obtained as the complement of the crisp representative of A in each level, as shown in Table I. The same idea is employed for union and intersection that yield the corresponding conjunction and disjunction of graded concepts, as shown in several columns in the table.

This way of operating has several implications. First, all the properties of the crisp case are kept, so RL-sets satisfy all the properties of Boolean algebras. This can be seen in Table I in the columns corresponding to the RLs for $A \wedge \neg A$ and $A \vee \neg A$ (at the same time, but not shown in the table, $A \wedge A = A \vee A = A$). This is related to the fact that, contrary to operations with fuzzy sets, operations with RLs are not truth-functional. Also, the translation of crisp operations like these to the case of graded concepts is unique and direct (think of set difference, implications, etc.). The price to pay is the necessity to maintain a collection of crisp representatives in memory, and the time needed for the computation (that can be alleviated via parallelization since operations are performed independently in each level).

The idea of performing operations on levels is not strange to fuzzy sets. It is well known that union and intersection performed via maximum and minimum, respectively, are equivalent to performing the union and intersection of the α-cuts in each level independently. This can be seen in Table II, where horizontal representations of fuzzy sets for concepts A, B, and $A \wedge B$ yield the same results obtained in Table I. However, the introduction of complement is radically different, as it breaks the ordinary nested relation between levels. This affects the results of operations, as it is evident when comparing both tables. It can also be observed in Table II how in this particular case properties such as the excluded middle are lost.

For those familiar with fuzzy sets, the RL resulting from the complement of a fuzzy set seems strange. How is it possible that, as you are less strict (lower levels), you have less elements? However, there is a rationale behind that: if at a certain level an object can be accepted as fulfiling A, then the object cannot be accepted as fulfiling $\neg A$ *in that same level*. Is the focus in the levels that changes the perspective. Note also that an object can appear in the representation of both A and $\neg A$, as in the case of fuzzy sets, but always in different levels.

C. A view of fuzzy sets from the Theory of RLs

In the theory of RLs it is common to measure the degree of fulfilment of properties as probability measures defined on the set of levels $(0,1]$, making use of Eq. (3). For instance, a degree of equality or inclusion between two RL-sets can be defined as the probability that in a level taken at random, both sets are equal or one of them is included in the other. As an example, in Table I, the probability that the RL-set for B is a subset of the RL-set for A is 0.5, since the inclusion only holds in the interval of levels $(0.5, 1]$. It is easy to see that the degree of equality is 0.2, as this holds in the interval of levels $(0.7, 0.9]$[2].

One remarkable property that can be measured in this way is the fulfilment of the property by a certain element of X. This provides us with a *degree of membership* of an object to an RL-set. When this measure is applied to every element of X for an RL ρ_A, the result can then be interpreted as a fuzzy set ν_A. Remarkably, since the set of crisp representatives in an RL is finite:

- If the levels of a RL-set ρ_A are the α-cuts of a fuzzy set μ_A with a finite set of membership values, then $\nu_A = \mu_A$. This reflect the ordinary one-to-one relation between fuzzy sets and their collection of α-cuts, in this case restricted to sets with a finite set of degrees Λ_A.
- In general it is possible to have others (even an infinite number of) RLs providing the same function ν.

As an example, the function $\nu_{A\wedge\neg B}$ for the corresponding RL-set in Table I is

$$\nu_{A\wedge\neg B} = 0.1/x_1 + 0.7/x_2 + 0.1/x_5.$$

Note that the α-cuts of $\nu_{A\wedge\neg B}$ do not match the RL for $A \wedge \neg B$, so we cannot use it for further operations following the RL philosophy. But this fuzzy set can serve as a *summary* of information about the membership of objects to an RL-set, complementing the information given by the RL-set itself about the output of the operations so far. For users, specially in the fuzzy community, this information can help in understanding what's going on. We'll come back to this point in Section IV.

TABLE III
TABLE

α	$\rho_e(\alpha)$	$\rho_{\lvert A\rvert}(\alpha)$	$\rho_{\lvert B\rvert}(\alpha)$	$\rho_{\lvert A\rvert+\lvert B\rvert}(\alpha)$
1	x_1	1	0	1
0.9	x_1	1	1	2
0.7	x_2	2	1	3
0.5	x_1	2	2	4
0.3	x_3	3	3	6
0.1	x_2	4	3	7

III. GRADUALITY IN ELEMENTS

One interesting feature of Eq. (2) is that we can change 2^X by any other set[3]. That is, we can consider graduality not only in subsets of a certain set X, but in elements of other sets. This is something that Eq. (1) cannot provide: fuzzy sets are always *sets*, with the particular feature of being affected by graduality.

The idea of *Gradual Element* was on the basis of the approach of Gradual Sets [1]. The column for ρ_e in Table III shows an example of gradual element taken from X (that is, $\rho_e : (0,1] \to X$). Again, the corresponding concept in the theory of RLs (RL-element) is similar, but limited to a finite set of crisp representatives (elements of X in this case). One particular property we can measure is membership. The membership of ρ_e to the RL-set for A in Table I is 1, since $\rho_e(\alpha) \in \rho_A(\alpha)\ \forall\alpha \in (0,1]$. It is easy to see that its membership to the RL-set for B is 0.6.

One particularly useful case of RL-element arises when they are defined on numbers [1], [9], [3]. This allows for instance to apply measuring to RL-sets, as in columns $\rho_{|A|}$ and $\rho_{|B|}$ in Table III, where the cardinality of the corresponding RL-sets in Table I is computed for each level. The last column in the table serves to show how the operations are performed by levels independently, the addition in this case. As in the crisp case, the operations of subtraction and division can be performed by considering RL-elements of the integers and rational numbers, real and complex when necessary, etc.

Note that RL-numbers, and Gradual Numbers in general, are gradual but not imprecise, to the extent that operations between gradual numbers may yield a crisp number (think for instance of $\rho_{|A|+|\neg A|}$, which is the crisp number 5 for A in Table I. On the contrary, the so-called *fuzzy numbers* are in fact fuzzy subsets of numbers, more properly fuzzy intervals, with imprecision increasing with operations.

[2]Gradual sets with an infinite amount of representatives require Lebesgue integration in general, as pointed out in [1].

[3]A study about the view of Gradual Sets defined on X as Gradual Elements of 2^X can be found in [8].

As a final remark, as was the case of sets, RL-numbers of every kind also keep the same algebraic structure as their crisp counterparts.

IV. The symbiosis: RL-systems

RL-systems provide an approach for solving problems involving graduality. The approach implies the use of both Fuzzy Sets and RLs. The ideas behind RL-systems are:

- Fuzzy Sets are employed as input. We consider that graduality comes fundamentally for graded concepts born in the human brain. Fuzzy Sets have proved to be an intuitive tool for representing such concepts, with many resources for elucidating membership functions.
- A finite amount of levels is considered. The particular levels may vary depending on the application at hand, representing the amount of levels we are able to distinguish (or less levels if they are enough for solving the problem).
- The fuzzy input is transformed into RLs by computing the α-cuts with the levels considered.
- Operations are performed in each level independently, including set operations, measuring and calculations with the obtained numbers.
- The output is provided as the resulting RL together with its corresponding fuzzy set ν in order to enhance the information provided to the user.

As an example, in [10], [11] these ideas are employed in order to perform flexible queries like *Find the students with high marks* in a relational database, using trapezoidal Fuzzy Sets for representing flexible restrictions like *high* in the previous query. Contrary to purely fuzzy solutions, all the properties of crisp queries are preserved, particularly the equivalence between different formulations.

Other applications in the literature include Data mining [12], [4] (see a review in [13]) and generalized quantification [14], among many others.

The RL-system approach is a good choice when at least one of the following hold:

- The solution is clear in the crisp case but it is not very clear how to extend it to the fuzzy case. In this case, the solution provided by RL-systems is to solve the crisp problem in each level directly.
- The problem to be solved involves working with mathematical objects other than sets, particularly numbers to be obtained by measurement.
- It is mandatory to keep properties of the crisp case that fuzzy approaches cannot provide.

V. Conclusions

We conceive the theory of RLs as another alternative to Fuzzy Set Theory for the same purposes: representing and managing graduality in real applications. The contributions of RLs are diverse: direct translation from the crisp to the gradual case, maintaining all the properties of the crisp case, and allowing to represent graduality beyond sets. The applications developed so far show the feasibility of the approach.

There are many research opportunities in this area. During the last 60 years, Fuzzy Sets have been applied in many fields, both from a mathematical/theoretical perspective and a more practical view. The use of RLs is worth exploring in those cases we have pointed out in the previous section. And RL-systems allow us to take benefit of the understandability of such a well-known representation tool as Fuzzy Sets are.

RLs also have their drawbacks, as we need to represent and to compute in different levels. Indexing and parallelizing can help with these aspects, as it has been also shown in practical applications.

And finally, no theory is better than the other in general. Both are useful and have an infinite research area to explore, and this is the path we will follow in the future.

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References

[1] D. Dubois and H. Prade, “Gradual elements in a fuzzy set,” *Soft Computing*, vol. 12, pp. 165–175, 2008.

[2] D. Sánchez, M. Delgado, and M. Vila, “A restriction level approach to the representation of imprecise properties,” in *Proceedings Int. Conference on Information Processing and Management of Uncertainty IPMU’08*, 2008, pp. 153–159.

[3] D. Sánchez, M. Delgado, M. Vila, and J. Chamorro-Martínez, “On a non-nested level-based representation of fuzziness,” *Fuzzy Sets and Systems*, vol. 192, no. 1, pp. 159–175, 2012.

[4] T. P. Martin and B. Azvine, “The X-mu approach: Fuzzy quantities, fuzzy arithmetic and fuzzy association rules,” in *IEEE Symposium on Foundations of Computational Intelligence, FOCI 2013, Singapore, Singapore, April 16-19, 2013*. IEEE, 2013, pp. 24–29.

[5] D. J. Lewis and T. P. Martin, “The X-μ approach: In theory and practice,” in *Information Processing and Management of Uncertainty in Knowledge-Based Systems - 15th International Conference, IPMU 2014, Montpellier, France, July 15-19, 2014, Proceedings, Part III*, ser. Communications in Computer and Information Science, A. Laurent, O. Strauss, B. Bouchon-Meunier, and R. R. Yager, Eds., vol. 444. Springer, 2014, pp. 406–415.

[6] T. P. Martin, “The X-mu representation of fuzzy sets,” *Soft Comput.*, vol. 19, no. 6, pp. 1497–1509, 2015.

[7] P. Jara, L. Merino, G. Navarro, and E. Santos, “On gradual sets, hesitant fuzzy sets, and representation theorems,” *IEEE Access*, vol. 12, pp. 111 158–111 168, 2024.

[8] J. M. García and P. Jara, “Gradual sets: An approach to fuzzy sets,” *Adv. Fuzzy Syst.*, vol. 2023, pp. 6 163 672:1–6 163 672:18, 2023.

[9] D. Sánchez, M. Delgado, and M. Vila, “RL-numbers: An alternative to fuzzy numbers for the representation of imprecise quantities,” in *FUZZ-IEEE 2008, Proceedings*. IEEE, 2008, pp. 2058–2065.

[10] P. Córdoba-Hidalgo, N. Marín, and D. Sánchez, “RL-instances: An alternative to conjunctive fuzzy sets of tuples for flexible querying in relational databases,” *Fuzzy Sets Syst.*, vol. 445, pp. 184–206, 2022.

[11] N. Marín, G. Rivas-Gervilla, and D. Sánchez, “Count-based flexible queries through RL-instances,” in *FUZZ-IEEE 2024, Yokohama, Japan, June 30 - July 5, Proceedings*. IEEE, 2024, pp. 1–8.

[12] D. Dubois and D. Sánchez, *Fuzzy Clustering based on Coverings*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2013, pp. 319–330.

[13] C. Molina, M. D. Ruiz, and J. Serrano, “Representation by levels: An alternative to fuzzy sets for fuzzy data mining,” *Fuzzy Sets Syst.*, vol. 401, pp. 113–132, 2020.

[14] M. D. Ruiz, D. Sánchez, and M. Delgado, “Level-based fuzzy generalized quantification,” *Fuzzy Sets Syst.*, vol. 345, pp. 24–40, 2018.

On the Future of Fuzzy Logic and Semantic Web Technologies

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***Abstract*—In this short paper, we discuss some thoughts about the past, present, and future of the research in the combination of fuzzy logic and Semantic Web technologies. We envision that this combination will be particularly fruitful in the context of Neurosymbolic Artificial Intelligence and point out some lessons already learnt that should be taken into account in the development of future technologies.**

***Index Terms*—fuzzy logic, ontologies, Description Logics, knowledge graphs**

I. Introduction

Since its birth in 1965, fuzzy logic has proven useful in many practical applications, from real-world problem solving (e.g., automatic control of industrial plants or conventional household appliances) to building generalisations of multiple areas within computer science, giving rise to fuzzy databases, fuzzy neural networks, fuzzy clustering, etc.

Nowadays, Semantic Web technologies have become a fundamental cornerstone of Knowledge Representation and Reasoning (KRR), one of the branches of Artificial Intelligence (AI), being a de facto standard to represent the relevant knowledge in any application domain. The objective of this short article is to highlight the impact (past, present, and future) of fuzzy logic in the field of Semantic Web technologies.

The remaining of this paper is organised as follows. After a brief background on Semantic Web technologies (Section II), we will discuss the role of fuzzy Semantic Web technologies in (Neurosymbolic) Artificial Intelligence (Section III) and point out several important lessons learnt that will be relevant in the future (Section IV).

II. Semantic Web technologies

Many modern approaches to represent knowledge were developed by the Semantic Web (SW) community, so they are often called Semantic Web (or semantic) technologies, even if they can be used beyond the web. Such technologies include ontologies and Knowledge Graphs (KGs).

- Ontologies are agreed schemas defining a common vocabulary (classes, relationships and axioms that must hold to keep the semantics of the knowledge). Ontologies are specified using a formal (logical-based) language.
- Ontology languages are typically based on Description Logics (DLs), a family of logics to represent structured knowledge with a very good trade-off between expressivity and computational complexity.
- KGs (such as Wikidata) are graph models used to represent data in terms of entities (nodes) and binary relations between them (edges). Typically, a KG is implemented as a set of triples, each of them linking an entity object with another entity or a literal value via a property. Usually, the classes and properties used in a KG are formally defined in an ontology.

However, since these technologies are not suitable enough to manage imprecision, as required by many – if not most – real-world domains, different fuzzy generalisations have been proposed, such as fuzzy ontologies [1], based on fuzzy Descriptive Logics [2], fuzzy knowledge graphs [3], etc. For example, in fuzzy ontologies, one can have fuzzy classes (defined using fuzzy sets), fuzzy properties (defined using fuzzy relations), fuzzy datatypes (defined using fuzzy membership functions), fuzzy axioms (with a partial degree of truth), and new methods to build complex classes (e.g., apart from the well-known Boolean connectors, it is possible to use an OWA aggregation).

III. Neurosymbolic Artificial Intelligence

While Semantic Web technologies are examples of symbolic AI, in recent years, we have witnessed a significant rise in subsymbolic AI. Thanks to the advances in machine learning, deep learning and Large Language Models (LLMs) have achieved remarkable results, but they still have several limitations. In this context, neurosymbolic AI [4] has been proposed, combining the advantages of both symbolic and subsymbolic AI. For example, Semantic Web technologies could be used to improve LLMs in different ways [5]:

- Ontologies can be used to detect logical contradictions between the background knowledge and the knowledge generated by the LLM, by solving a consistency test.
- LLMs are black-box models that often cannot explain how their conclusions were derived, but ontologies can improve the explainability by using logical reasoning to justify their inferences.
- Using knowledge injection, LLMs can take advantage of the information stored in a knowledge graph.

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In these new hybrid models, it makes a lot of sense to incorporate fuzzy logic as well, to manage imprecision and vagueness. For example:

- In some cases, the law of excluded middle does not hold: knowing both a piece of information and its negation does not necessarily mean a logical contradiction. Such behaviour can be modelled with fuzzy DLs.
- Fuzzy logic is very useful to build explainable intelligent systems, as linguistic labels are very good for summarising information.
- Knowledge injection might include linguistic variables and fuzzy rules, which appear in some existing knowledge bases.

Years ago, most of the attention in the field of Semantic Web technologies was dedicated to ontologies. Nowadays, knowledge-based graphs are probably more popular. For sure, in the future, new semantic technologies will emerge, and new fuzzy extensions of them will be convenient. Therefore, it is important not to forget what we have already learnt when building such future fuzzy technologies.

IV. Some lessons learnt

Fuzzy logic vs. crisp logic: We do not need a fuzzy version of everything: we should not fall into the temptation of fuzzifying each new technology for its own sake. Instead, we should be guided by real-world applications: whenever we deal with real-world knowledge, it is quite likely that the need (or convenience) of using fuzzy logic to manage the vagueness inherent in natural language will naturally emerge.

Fuzzy logic in wide sense vs. fuzzy logic in a narrow sense: Lotfi A. Zadeh distinguished between fuzzy logic in a narrow sense (understood as a logical system) and fuzzy logic in a wide or broad sense (with a more general meaning, as the theory of fuzzy sets). Both are important and deserve attention, as each has its own applications. In the case of Semantic Web technologies, both the study of the complexity of fuzzy DLs [2], [6] and the development of languages to encode fuzzy ontologies [7] have been important.

Fuzzy logic vs fuzzy logics: We frequently use the term "fuzzy logic" in singular, but sometimes we should better use it in plural. While in the classical case there is, for example, a unique DL called $\mathcal{ALC}$, in the fuzzy case there are several fuzzy extensions, as there are several degrees of freedom: different fuzzy operators can be used to define the semantics of the logic, the syntax of the axioms can be extended to the fuzzy case or not, etc.

Clearly, the choice of fuzzy operators is significant. Gödel $\mathcal{ALC}$ and Łukasiewicz $\mathcal{ALC}$, defined from the minimum t-norm and the Łukasiewicz t-norm, respectively, have different logical properties. In fact, while with some fuzzy operators the complexity of a DL does not increase, other fuzzy operators make the logic undecidable [6]. Furthermore, while in the classical case $\mathcal{ALC}$ and $\mathcal{ALCU}$ DLs are equivalent, since union can be defined from negation and conjunction, in the fuzzy case this is not always the case [2].

Fuzzy extension vs. fuzzy creation: It is always possible to propose a new language to represent or query fuzzy knowledge. However, in many cases, it could be preferable to reuse existing standard languages in order to promote user adoption of the novel technology. This can be achieved by building a fuzzy layer on top of the standard language. Furthermore, even if the user has a classical knowledge base (e.g., an ontology or a knowledge graph), it is also possible to solve flexible queries (including imprecise terms) over it, so that the representation language remains the standard one and only the mechanism to query the knowledge base needs to be modified [3], [8].

Similarly, to address the reasoning, we can develop new algorithms or reduce the problem to already known algorithms. On the one hand, fuzzy DL reasoning can sometimes be reduced to classical reasoning, avoiding the need to develop a dedicated reasoner at the cost of increasing the size of the knowledge base [8]. On the other hand, the development of new algorithms avoids such an increase but requires new implementations. For example, the fuzzyDL reasoner was built completely from scratch [9]. New algorithms include extensions of existing ones (e.g., fuzzy extensions of the classical tableaux algorithms [10]) as well as novel solutions (e.g., combinations of tableaux and linear programming [11]). Existing families of reasoning algorithms for fuzzy DLs support different languages and have different properties. Again, applications will guide the user in choosing the most appropriate family and its software implementation.

References

[1] U. Straccia, *Foundations of Fuzzy Logic and Semantic Web Languages*, ser. CRC Studies in Informatics Series. Chapman & Hall, 2013.

[2] F. Bobillo, M. Cerami, F. Esteva, À. García-Cerdaña, R. Peñaloza, and U. Straccia, "Fuzzy description logics," in *Handbook of Mathematical Fuzzy Logic Volume III*, ser. Studies in Logic, Mathematical Logic and Foundations, P. Cintula, C. Fermüller, and C. Noguera, Eds. College Publications, 2015, vol. 58, ch. XVI, pp. 1105–1181.

[3] J. F. Yagüe, I. Huitzil, C. Bobed, and F. Bobillo, "FUKG: Answering flexible queries over knowledge graphs," *The Electronic Library*, vol. 42, no. 3, pp. 368–392, 2023.

[4] P. Hitzler, F. Bianchi, M. Ebrahimi, and M. K. Sarker, "Neural-symbolic integration and the semantic web," *Semantic Web*, vol. 11, no. 1, pp. 3–11, 2020.

[5] F. Bobillo, E. Mena, J. Gracia, and C. Bobed, "HAIKU: Hybrid artificial intelligence on knowledge and linguistics," in *Actas de las XXIX Jornadas de Ingeniería del Software y Bases de Datos (JISBD 2025)*, 2025.

[6] F. Baader, S. Borgwardt, and R. Peñaloza, "Decidability and complexity of fuzzy description logics," *Künstliche Intelligenz*, vol. 31, pp. 85–90, 2017.

[7] F. Bobillo and U. Straccia, "Fuzzy ontology representation using OWL 2," *International Journal of Approximate Reasoning*, vol. 52, no. 7, pp. 1073–1094, 2011.

[8] F. Bobillo, "The role of crisp elements in fuzzy ontologies: The case of fuzzy OWL 2 EL," *IEEE Transactions on Fuzzy Systems*, vol. 24, pp. 1193–1209, 2016.

[9] F. Bobillo and U. Straccia, "The fuzzy ontology reasoner fuzzyDL," *Knowledge-Based Systems*, vol. 95, pp. 12–34, 2016.

[10] G. Stoilos and G. Stamou, "Reasoning with fuzzy extensions of OWL and OWL 2," *Knowledge and Information Systems*, vol. 40, pp. 205—-242, 2014.

[11] U. Straccia, "Description logics with fuzzy concrete domains," in *Proceedings of the 21st Conference on Uncertainty in Artificial Intelligence (UAI 2005)*. AUAI Press, 2005, pp. 559–567.

Potential New Research Directions for Fuzzy Systems

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***Abstract*—Fuzzy sets and systems can be used in intelligent control, decision making, and machine learning. This short article briefly reviews some of our recent works on efficient optimizing TSK fuzzy systems for machine learning, and fuzzy sets for extending brain-computer interface algorithms from classification to regress. It also points out some potential new research directions.**

***Index Terms*—Fuzzy system, residual network, attention, brain-computer interface, transfer learning**

I. Introduction

This year marks the 60th anniversary of fuzzy sets [1]. According to the author's preliminary survey, Zadeh's 1965 "Fuzzy Sets" paper [1] is the 5th most cited machine learning paper on Google Scholar, after four classic papers that proposed residual network (ResNet) [2], Adam optimizer [3], attention mechanism [4], and AlexNet [5]. Notably, the "Fuzzy Sets" paper is the only single-authored paper among the top 5.

My main research interests include fuzzy logic, brain-computer interface, and machine learning. Next, I'll briefly introduce some of our recent works on fuzzy logic, and potential new directions.

II. Fuzzy Systems for Machine Learning

Recently, we identified the functional equivalences/similarities between TSK fuzzy systems and several classic machine learning models [6], including neural networks, mixture-of-experts, classification and regression tree (CART), and stacking ensemble learning. Based on these equivalences/similarities, we extended/adapted some effective techniques, e.g., mini-batch gradient descent, batch normalization, dropOut, ReLU, and Adam, from the training of (deep) neural networks to the training of TSK fuzzy systems [7]–[11], and achieved promising performance. Particularly, they enhanced TSK fuzzy systems' capability to handle big data and high-dimensional features.

More novel architectures and training algorithms for deep neural networks may also be extended/adapted to the training of fuzzy systems, e.g.,

1) *ResNet* [2]. The most distinguishing innovation of ResNet is the skip-layer connection, enabling the construction and training of very deep neural networks. Whereas skip-layer connections were first proposed for neural networks in 2016, similar idea existed in TSK fuzzy systems since their born: In the popular adaptive-network-based fuzzy inference system (ANFIS) [12] shown in Fig. 1, the inputs $(x_1, x_2, ..., x_d)$ enter the first layer, and also the 4th layer directly (i.e., they skip the 2nd and 3rd layers). The numerous new architecture variants and/or training algorithms for ResNet may have their counterparts for TSK fuzzy systems, which remain to be explored.

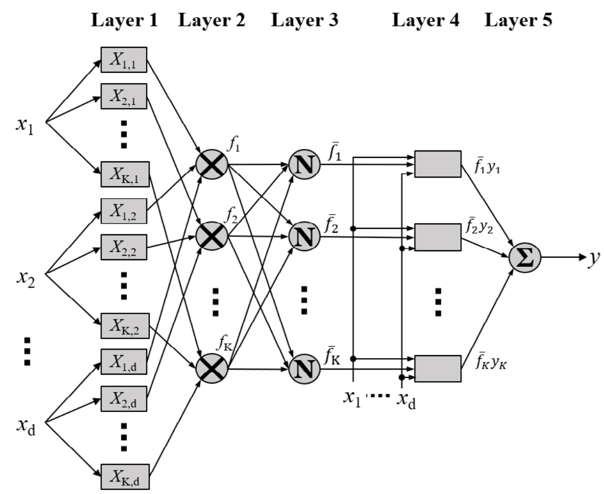

Fig. 1. The 5-laye ANFIS representation of a TSK fuzzy system.

2) *Attention* [4]. The attention mechanism has demonstrated promising performance in deep learning when dealing with computer vision and natural language processing tasks. It includes three main components: a query Q, a key K, and a value V, which are combined using the following equation:

$$Z = \text{softmax}\left(\frac{QK^\top}{\sqrt{d}}\right)V, \tag{1}$$

where d is the feature dimensionality.

For a TSK fuzzy system, we can view the process for computing the firing levels of the rules as softmax$\left(\frac{QK^\top}{\sqrt{d}}\right)$, and the rule consequents as V; thus, a TSK fuzzy system actually uses the attention mechanism. The numerous new architecture variants and/or training algorithms for attention in neural networks may have their counterparts for TSK fuzzy systems, which also remain to be explored.

III. Fuzzy Sets for Brain-Computer Interfaces

A brain-computer interface (BCI) is a direct communication pathway between a user's brain and an external device, which can be used to map, assist, augment, or repair the cognitive and/or sensory-motor functions of the brain [13]. Due to its convenience and low cost, electroencephalogram (EEG) is the most popular input for non-invasive BCIs.

A major challenge in EEGb-based BCIs is to accommodate individual differences, or inter-subject variability [14], and hence to expedite the calibration for a new user. Transfer learning [15], which uses data/knowledge from auxiliary (source) subjects to facilitate the learning for a new subject, has been extensively used in BCI calibration.

There are currently numerous algorithms for classification problems in EEG-based BCIs, but not for regression problems. To avoid the burden of developing algorithms for BCI regression problems from the scratch, we have proposed some fuzzy set based approaches to extend classic signal process and machine learning algorithms from classification to regression, and demonstrated promising performance [16], [17]. The basic idea is to use fuzzy sets to construct fuzzy classes, so that class conditional calculations can be performed for regression problems.

The above idea is particularly promising for extending transfer learning algorithms from classification to regression, as the former depend heavily on aligning the class conditional probabilities between the source subjects and the target subject. We believe lots of interesting new results could be developed.

IV. Conclusions

This short article briefly reviews some of our recent works on fuzzy systems for machine learning, and fuzzy sets for brain-computer interfaces. It also points out some potential new research directions.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, pp. 338–353, 1965.

[2] K. He, X. Zhang, S. Ren, and J. Sun, "Deep residual learning for image recognition," in *Proc. IEEE Conf. on Computer Vision and Pattern Recognition*, Las Vegas, NV, June 2016, pp. 770–778.

[3] D. P. Kingma and J. Ba, "Adam: A method for stochastic optimization," in *Proc. 3rd Int'l Conf. on Learning Representations*, San Diego, CA, May 2015.

[4] A. Vaswani, N. Shazeer, N. Parmar, J. Uszkoreit, L. Jones, A. N. Gomez, L. Kaiser, and I. Polosukhin, "Attention is all you need," in *Proc. Advances in Neural Information Processing Systems*, Long Beach, CA, Dec. 2017.

[5] A. Krizhevsky, I. Sutskever, and G. E. Hinton, "ImageNet classification with deep convolutional neural networks," in *Proc. Advances in Neural Information Processing Systems*, Lake Tahoe, NV, December 2012, pp. 1097–1105.

[6] D. Wu, C.-T. Lin, J. Huang, and Z. Zeng, "On the functional equivalence of TSK fuzzy systems to neural networks, mixture of experts, CART, and stacking ensemble regression," *IEEE Trans. on Fuzzy Systems*, vol. 28, no. 10, pp. 2570–2580, 2020.

[7] D. Wu, Y. Yuan, J. Huang, and Y. Tan, "Optimize TSK fuzzy systems for regression problems: Mini-batch gradient descent with regularization, DropRule and AdaBound (MBGD-RDA)," *IEEE Trans. on Fuzzy Systems*, vol. 28, no. 5, pp. 1003–1015, 2020.

[8] Y. Cui, J. Huang, and D. Wu, "Optimize TSK fuzzy systems for classification problems: Mini-batch gradient descent with uniform regularization and batch normalization," *IEEE Trans. on Fuzzy Systems*, vol. 28, no. 12, pp. 3065–3075, 2020.

[9] Z. Shi, D. Wu, C. Guo, C. Zhao, Y. Cui, and F.-Y. Wang, "FCM-RDpA: TSK fuzzy regression model construction using fuzzy c-means clustering, regularization, DropRule, and Powerball AdaBelief," *Information Sciences*, vol. 574, pp. 490–504, 2021.

[10] Y. Cui, D. Wu, and Y. Xu, "Curse of dimensionality for TSK fuzzy neural networks: Explanation and solutions," in *Proc. IEEE Int'l Joint Conf. on Neural Networks*, Shenzhen, China, Jul. 2021.

[11] Y. Cui, Y. Xu, R. Peng, and D. Wu, "Layer normalization for TSK fuzzy system optimization in regression problems," *IEEE Trans. on Fuzzy Systems*, vol. 31, no. 1, pp. 254–264, 2023.

[12] J. R. Jang, "ANFIS: adaptive-network-based fuzzy inference system," *IEEE Trans. on Systems, Man, and Cybernetics*, vol. 23, no. 3, pp. 665–685, 1993.

[13] M. O. Krucoff, S. Rahimpour, M. W. Slutzky, V. R. Edgerton, and D. A. Turner, "Enhancing nervous system recovery through neurobiologics, neural interface training, and neurorehabilitation," *Frontiers in Neuroscience*, vol. 10, p. 584, 2016.

[14] W.-L. Z. Z. L. Wu, Xun and B.-L. Lu, "Investigating eeg-based functional connectivity patterns for multimodal emotion recognition," *Journal of neural engineering*, vol. 19, no. 1, p. 016012, 2022.

[15] D. Wu, Y. Xu, and B.-L. Lu, "Transfer learning for EEG-based brain-computer interfaces: A review of progress made since 2016," *IEEE Trans. on Cognitive and Developmental Systems*, vol. 14, no. 1, pp. 4–19, 2022.

[16] D. Wu, V. J. Lawhern, S. Gordon, B. J. Lance, and C.-T. Lin, "Driver drowsiness estimation from EEG signals using online weighted adaptation regularization for regression (OwARR)," *IEEE Trans. on Fuzzy Systems*, vol. 25, no. 6, pp. 1522–1535, 2017.

[17] D. Wu, J.-T. King, C.-H. Chuang, C.-T. Lin, and T.-P. Jung, "Spatial filtering for EEG-based regression problems in brain-computer interface (BCI)," *IEEE Trans. on Fuzzy Systems*, vol. 26, no. 2, pp. 771–781, 2018.

Fuzzy Logic and the Future of Human-Centric AI

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60 Years of Fuzzy Logic
– Shaping Intelligent Systems

***Abstract*—At the time we celebrate 60 years since fuzzy logic, we are also faced with one of the pressing challenges of our time: how artificial intelligence evolves not as a replacement for human beings but as a friendly partner. At a time of increasing automation, algorithms, and efficient systems, we ask: what kind of intelligence do we need? We postulate that Fuzzy Logic is what AI systems require to behave more like assistants, collaborators, or colleagues rather than machines.**

***Index Terms*—Fuzzy Sets and Systems, Artificial Intelligence, Nuance Intelligence**

I. Introduction

Fuzzy Sets and Systems – Happy 60th Birthday! It is almost nothing in terms of a celestial body but quite a bit in the scale of a human. However, in the case of an idea, theory, or mathematical concept, it may be the right time to discover its ultimate purpose, the time to reveal its influence.

Examining the brief history of fuzzy sets and systems reveals their tremendous success. As a very new concept, it has generated a lot of interest in both theoretical and applied areas. The development of fuzzy control, possibility theory, aggregation operators, and fuzzy expert systems are just a few examples of the most recognizable achievements.

At 60, fuzzy sets and systems have witnessed an enormous and successful development of AI-related technologies – from deep neural networks in pattern recognition to large foundation models in the areas of natural language and image processing. Yet, human interaction with machines is problematic and intricate. The current AI systems, having impressive capabilities, struggle with fundamental limitations. Large language models produce overconfident predictions without measures of uncertainty. Neural networks operate as black boxes, lacking explanations when regulations and ethics mandate transparency. Binary decision-making systems fail to capture the nuanced reasoning that characterizes human intelligence.

II. Opportunity

The original concept of fuzzy sets and systems, as envisioned by Lotfi A. Zadeh [1]–[3], aimed to address the issue of interaction between humans, who are inherently imprecise and vague entities, and computers, which are precise and well-defined. It appears that there is an opportunity to address challenges facing modern AI, i.e., uncertainty quantification, interpretability issues, and the need for human-aligned reasoning. That all aligns perfectly with the solutions **fuzzy logic** has to offer. As AI looks beyond deterministic out-puts toward context-aware, explainable, and ethical systems, fuzzy logic becomes fundamental not as a competitor to neural methods but as a **complementary reasoning layer**.

In various fields, including healthcare, education, personalized AI assistants, and autonomous decision-making, we require systems that not only process data but also weigh conflicting, incomplete, and qualitative inputs just as people do. Fuzzy systems can enable AI to act in ways aligned with human expectations, allowing it to tolerate ambiguity. In this case, fuzzy logic becomes a vital component. It will enable AI systems to behave more like assistants, collaborators, or colleagues rather than machines.

III. Nuance Intelligence

This interaction of AI and Fuzzy Logic leads to

> **Nuance Intelligence** – *systems capable of subtle reasoning, contextual adaptation, and human-like understanding.*

Unlike traditional AI, which prioritizes precision, Nuance Intelligence focuses on ambiguity, degrees of truth, and fuzzy boundaries. These are aspects that humans are familiar with and care about. It is not about automation or replacements but about building systems that collaborate, explain, and work for humans using human measures and criteria.

Consider a medical diagnosis system powered by Nuance Intelligence:

Traditional AI: *Patient has 87% probability of condition X*

Nuance Intelligence: *Patient shows moderate symptoms (0.7 confidence) with borderline test results (0.6 confidence), suggesting condition X with qualified certainty. Age factor is somewhat elevated (0.8), while lifestyle factors are moderately protective (0.5).*

The fuzzy-based approach not only provides a decision but explains the reasoning process, communicates uncertainty levels, and offers insight into contributing factors, all in human-understandable terms. Additionally, Nuance Intelligence addresses the need for explainability. Systems built with such intelligence have explainability built into their core architecture. The linguistic variables and fuzzy rule-based structures provide natural explanations. When a system indicates that the temperature is moderately high, humans understand both the measurement and its implications.

IV. Conclusion

All this is fuzzy reasoning at work: capturing and expressing degrees of belief, possibility, relevance, or safety.

Nuance Intelligence – Fuzzy by Design:
With Us. For Us. Never instead of Us.

Happy Birthday, Fuzzy Sets and Systems!

Wishing you to become easily recognizable,
widely adopted and justly successful

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] ——, "Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic," *Fuzzy Sets and Systems*, vol. 90, no. 2, pp. 111–127, 1997.

[3] ——, "Is there a need for fuzzy logic?" *Information Sciences*, vol. 178, pp. 2751–2779, 2008.

Operators in Finite Chains and Their Extensions in Fuzzy Logic

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***Abstract*—Research on operators in finite chains has been a field of constant study in the framework of fuzzy logic. Therefore, in this short paper, we want to show the importance of this active field of research, some lines of investigation that have arisen, as well as new challenges that remain to be done.**

***Index Terms*—Discrete fuzzy numbers, finite chains, admissible order, qualitative reasoning, t-norms, aggregation operators, cardinality.**

I. Introduction

In today's world, where information is often presented in a massive, imprecise, and distorted manner, it is particularly important to have tools that facilitate its understanding and processing. Fuzzy logic [1] provides an appropriate framework for this purpose. Although the unit interval has traditionally been used as the standard domain for modeling and defining operators, when dealing with ordinal or qualitative information [2], [3], more suitable structures are required, such as the finite chain $L_n = \{0, 1, \ldots, n\}$, which models a totally ordered set of $n+1$ elements. The foundation for modeling qualitative information on finite chains and defining operators within this framework was established by Godo and Sierra [4]. In earlier work [5], linguistic terms like *Impossible*, *Almost possible*, and *Sure* were modeled using fuzzy sets over the unit interval. However, two main drawbacks were identified: the difficulty experts face in translating linguistic labels into precise membership functions, and the inconsistency in representations provided by different experts. To address these issues, the authors proposed modeling labels directly using a finite chain $E = \{E_0, E_1, \ldots, E_n\}$ equipped with a total order $\preceq$, and redefined classical fuzzy operators—such as negations, t-norms, and t-conorms—to operate on this structure. It was later recognized that E could be naturally represented using L_n, allowing direct mapping of linguistic scales like the one in [5] (impossible, almost impossible, slightly possible, ..., almost sure, sure) onto L_8.

In this framework, significant progress has been made in three main areas:

i) theoretical study of operators on L_n [3], [6]–[14],
ii) their enumeration [10], [15]–[21],
iii) and their application to decision making, consensus, and image processing [2], [22]–[24]

The elements of a finite chain, interpreted as linguistic expressions of an expert, can result in very rigid evaluations in a

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decision-making problem. For this reason, the set of discrete fuzzy numbers with values in a finite chain L_n, denoted by $\mathcal{A}_1^{L_n}$ [22], offers a more flexible framework for qualitative reasoning. In this environment, different lines of research have been put forward:

- Linguistic models based on finite chains ([25], [26]),
- Theoretical investigation base on operators defined on L_n [22], [27] (t-norms, t-conorms, uninorms, implications,...)
- Applications.

The remainder of this paper is organized as follows. In Section II, we introduce the finite chain L_n and review the main classes of operators defined on it. Section III presents the formal definition of discrete fuzzy numbers over L_n and some basic concepts on this framework. Finally, in Section IV, we outline open questions and future research directions about these topics.

II. Operators on the finite chain L_n

In the realm of qualitative information processing, the finite chain $L_n = \{0, 1, \dots, n\}$ serves as a fundamental structure for modeling ordinal and linguistic information. While classical fuzzy logic typically operates on the unit interval $[0, 1]$, many real-world applications—particularly those involving human perception or expert knowledge—require reasoning over discrete, ordered domains. In this context, the definition and study of suitable operators on L_n become essential to perform logical and computational tasks such as negation, conjunction, disjunction, and aggregation.

Let us consider $L_n = \{0, 1, \dots, n\}$, where $n \in \mathbb{N}$. The elements of L_n are interpreted as qualitative labels that can represent degrees of truth, confidence, preference, or any other kind of ordered linguistic term. Since L_n is totally ordered, it provides a natural environment for defining discrete analogs of fuzzy logic operators.

The most commonly studied classes of operators on L_n include:

- Fuzzy negations: Functions $N : L_n \to L_n$ that reverse the order of truth values.
- Triangular norms (t-norms): Functions $T : L_n \times L_n \to L_n$ that generalize the logical conjunction.
- Triangular conorms (t-conorms): Functions $S : L_n \times L_n \to L_n$ that generalize the logical disjunction.
- Aggregation functions: Functions $A : L_n^k \to L_n$ that combine multiple inputs into a single output, preserving monotonicity.
- Implication functions: Functions $I : L_n \times L_n \to L_n$ that generalizes the notion of crisp implication from classical logic to the finite chain $L_n = \{0, 1, \dots, n\}$.

All these operators must satisfy certain axioms adapted from their continuous counterparts, while respecting the discrete nature of L_n.

III. Discrete Fuzzy Numbers and Admissible Orders

Building upon the structure of the finite chain $L_n = \{0, 1, \dots, n\}$ and the operators defined on it, we now introduce discrete fuzzy numbers (DFNs), which provide a formal framework for modeling qualitative and imprecise information in decision-making contexts.

A *discrete fuzzy number* (DFN) is a fuzzy set $A : \mathbb{R} \to [0, 1]$ with finite support $\operatorname{supp}(A) = \{x_1, x_2, \dots, x_k\} \subset L_n$, where $x_1 < x_2 < \cdots < x_k$, and membership degrees satisfying the following properties (see Voxman, 2001 [28]):

1) There exist indices $s \leq t$ such that $A(x_i) = 1$ for all $i \in \{s, \dots, t\}$. This set is called the *core* of A.
2) The membership function increases up to the core: $A(x_i) \leq A(x_j)$ for all $i \leq j \leq s$.
3) The membership function decreases after the core: $A(x_i) \geq A(x_j)$ for all $t \leq i \leq j$.

We denote by $\mathcal{D}L_n$ the set of all DFNs whose support lies within L_n, and by $\mathcal{A}_1^{L_n} \subseteq \mathcal{D}L_n$ the subset of DFNs whose support forms a closed subinterval of L_n, i.e., of the form $[i, j]$ with $i, j \in L_n$ and $i \leq j$. This subclass plays a key role in modeling linguistic expressions in decision-making problems [25].

Given a DFN $A \in \mathcal{A}_1^{L_n}$, its *$\alpha$-cut at level* $\alpha \in (0, 1]$ is defined as:

$$A_\alpha = \{x \in L_n \mid A(x) \geq \alpha\}.$$

The α-cuts are intervals in L_n, and they are used to compare DFNs through their structural components.

To allow for more structured and interpretable representations, especially in computational models, we consider DFNs whose membership values are restricted to a finite scale $Y_m = \{y_1 = 0, y_2, \dots, y_m = 1\}$, where $0 = y_1 < y_2 < \cdots < y_{m-1} < y_m = 1$. We denote by $\mathcal{A}_1^{L_n \times Y_m}$ the set of DFNs over L_n whose membership values belong to Y_m. This restriction enhances the usability of DFNs in real-world applications where experts express their opinions using a predefined finite linguistic scale. An important property of this class is that it contains only a finite number of elements. Specifically, as shown in Mir-Fuentes et al. [26], the total number of such DFNs is given by:

$$\left|\mathcal{A}_1^{L_n \times Y_m}\right| = \binom{n + 2m - 2}{2m - 2}.$$

This result follows from modeling each DFN as a pair consisting of a support interval $[i, j] \subseteq L_n$ and a membership profile over that interval, which must satisfy increasing-decreasing behavior with respect to the Y_m-valued membership degrees.

This finiteness allows us to treat $\mathcal{A}_1^{L_n \times Y_m}$ as a finite chain, which is essential for defining computable operations and orders.

IV. Open Questions and Future Work

This section sets out some possible lines of research.

A. Finite Chains

- To study the cardinality of discrete implication functions satisfying certain properties (Exchange Principle, the law of importation, with respect to a discrete t-norm T, etc.)

- Investigate the cardinality of discrete conjunctions, disjunctions, and implications, focusing on those that are smooth, including an asymptotic analysis.

B. Discrete fuzzy numbers defined on a finite chain

- Generalization of operators in the set $\mathcal{A}_1^{L_n \times Y_m}$ based on operators defined on the finite chain L_n.
- Because the cardinality of $\mathcal{A}_1^{L_n \times Y_m}$ increases very rapidly, efficient algorithms are required to generalize the operators discussed in the previous point.
- Develop new linguistic models based on discrete fuzzy numbers to enable the accurate representation of expert opinions. For instance, the linguistic model based on Z-numbers [29] (this model provides a more appropriate framework for capturing both the fuzziness and reliability of expert knowledge).

References

[1] L. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] R. R. Yager, "An approach to ordinal decision making," *International Journal of Approximate Reasoning*, vol. 12, pp. 237–261, 1995.

[3] G. Mayor and J. Torrens, "Triangular norms on discrete settings," in *Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms*, E. P. Klement and R. Mesiar, Eds. Elsevier, 2005, pp. 189–230.

[4] L. Godo and C. Sierra, "A new approach to connective generation in the framework of expert systems using fuzzy logic," in *The Eighteenth International Symposium on Multiple-Valued Logic*. IEEE Computer Society Press, 1988, pp. 157–162.

[5] L. Godo, R. López de Mántaras, C. Sierra, and A. Verdaguer, "Managing Linguistically Expressed Uncertainty in MILORD Application to Medical Diagnosis," *AI Communications*, vol. 1, pp. 14–31, 1988.

[6] J. L. Marichal and R. Mesiar, "Aggregation on finite ordinal scales by scale independent functions," *Order*, vol. 21, pp. 155–180, 2004.

[7] G. Mayor and J. Torrens, "On a class of operators for expert systems," *International Journal of Intelligent Systems*, vol. 8, pp. 771–778, 1993.

[8] M. Nachtegael and E. Kerre, "Fuzzy logical operators on finite chains," *International Journal of General Systems*, vol. 29, pp. 29–52, 2000.

[9] M. Mas, M. Monserrat, and J. Torrens, "On two types of discrete implications," *International Journal of Approximate Reasoning*, vol. 40, pp. 262–279, 2005.

[10] G. Mayor, J. Suñer, and J. Torrens, "Copula-like operations on finite settings," *IEEE Transactions on Fuzzy Systems*, vol. 13, pp. 468–477, 2005.

[11] B. De Baets, J. Fodor, D. Ruiz-Aguilera, and J. Torrens, "Idempotent uninorms on finite ordinal scales," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol. 17, pp. 1–14, 2009.

[12] M. Mas, M. Monserrat, and J. Torrens, "The law of importation for discrete implications," *Information Sciences*, vol. 179, pp. 4208–4218, 2009.

[13] ——, "Kernel aggregation functions on finite scales. Constructions from their marginals," *Fuzzy Sets and Systems*, vol. 241, pp. 27–40, 2014.

[14] M. Munar, S. Massanet, and D. Ruiz-Aguilera, "A review on logical connectives defined on finite chains," *Fuzzy Sets and Systems*, vol. 462, p. 108469, 2023.

[15] I. Aguiló, J. Suñer, and J. Torrens, "Matrix representation of discrete quasi-copulas," *Fuzzy Sets and Systems*, vol. 159, pp. 1658–1672, 2008.

[16] C. Bejines and M. Navara, "The Fibonacci sequence in the description of maximal discrete Archimedean t-norms," *Fuzzy Sets and Systems*, vol. 451, pp. 94–112, 2022.

[17] C. Bejines and M. Ojeda-Hernández, "Counting semicopulas on finite structures," *Fuzzy Sets and Systems*, vol. 462, p. 108405, 2023.

[18] M. Munar, S. Massanet, and D. Ruiz-Aguilera, "On the cardinality of some families of discrete connectives," *Information Sciences*, vol. 621, pp. 708–728, 2023.

[19] ——, "A study on the cardinality of some families of discrete operators through alternating sign matrices," *Information Sciences*, vol. 639, p. 118571, 2023.

[20] M. Munar, M. Couceiro, S. Massanet, and D. Ruiz-Aguilera, "A survey on the enumeration of classes of logical connectives and aggregation functions defined on a finite chain, with new results," *Fuzzy Sets and Systems*, vol. 490, p. 109023, 2024.

[21] M. Munar, A. Mir, S. Massanet, and D. Ruiz-Aguilera, "An analysis of the asymptotic behavior of the cardinality of some classes of logical connectives and aggregation functions defined on finite chains," *Computational and Applied Mathematics*, vol. 44, p. 232, 2025.

[22] J. V. Riera and J. Torrens, "Aggregation of subjective evaluations based on discrete fuzzy numbers," *Fuzzy Sets and Systems*, vol. 191, pp. 21–40, 2012.

[23] M. González-Hidalgo and S. Massanet, "A fuzzy mathematical morphology based on discrete t-norms: fundamentals and applications to image processing," *Soft Computing*, vol. 18, pp. 2297–2311, 2014.

[24] M. González-Hidalgo, M. Munar, P. Bibiloni, G. Moyà-Alcover, A. Craus-Miguel, and J. Segura-Sampedro, "Detection of infected wounds in abdominal surgery images using fuzzy logic and fuzzy sets," in *2019 International Conference on Wireless and Mobile Computing, Networking and Communications (WiMob)*. IEEE Computer Society, 2019, pp. 99–106.

[25] S. Massanet, J. V. Riera, J. Torrens, and E. Herrera-Viedma, "A new linguistic computational model based on discrete fuzzy numbers for computing with words," *Inf. Sci.*, vol. 258, p. 277–290, Feb. 2014.

[26] A. Mir-Fuentes, L. De Miguel, S. Massanet, A. Mir, and J. Riera, "On a total order on the set of z-numbers based on discrete fuzzy numbers," *Computational and Applied Mathematics*, vol. 43, no. 5, pp. 1–19, 2024.

[27] M. González Hidalgo, S. Massanet, A. Torres, J. Riera, and L. Miguel, "On fuzzy implication functions based on admissible orders on the set of discrete fuzzy numbers," *International Journal of Computational Intelligence Systems*, vol. 18, 06 2025.

[28] W. Voxman, "Canonical representations of discrete fuzzy numbers," *Fuzzy Sets and Systems*, vol. 54, no. 3, pp. 457–466, 2001.

[29] L. A. Zadeh, "A note on z-numbers," *Information Sciences*, vol. 181, no. 14, pp. 2923–2932, 2011.

Illustrating Six Decades of Fuzzy Logic: From Zadeh's Seminal Work to Contemporary Applications

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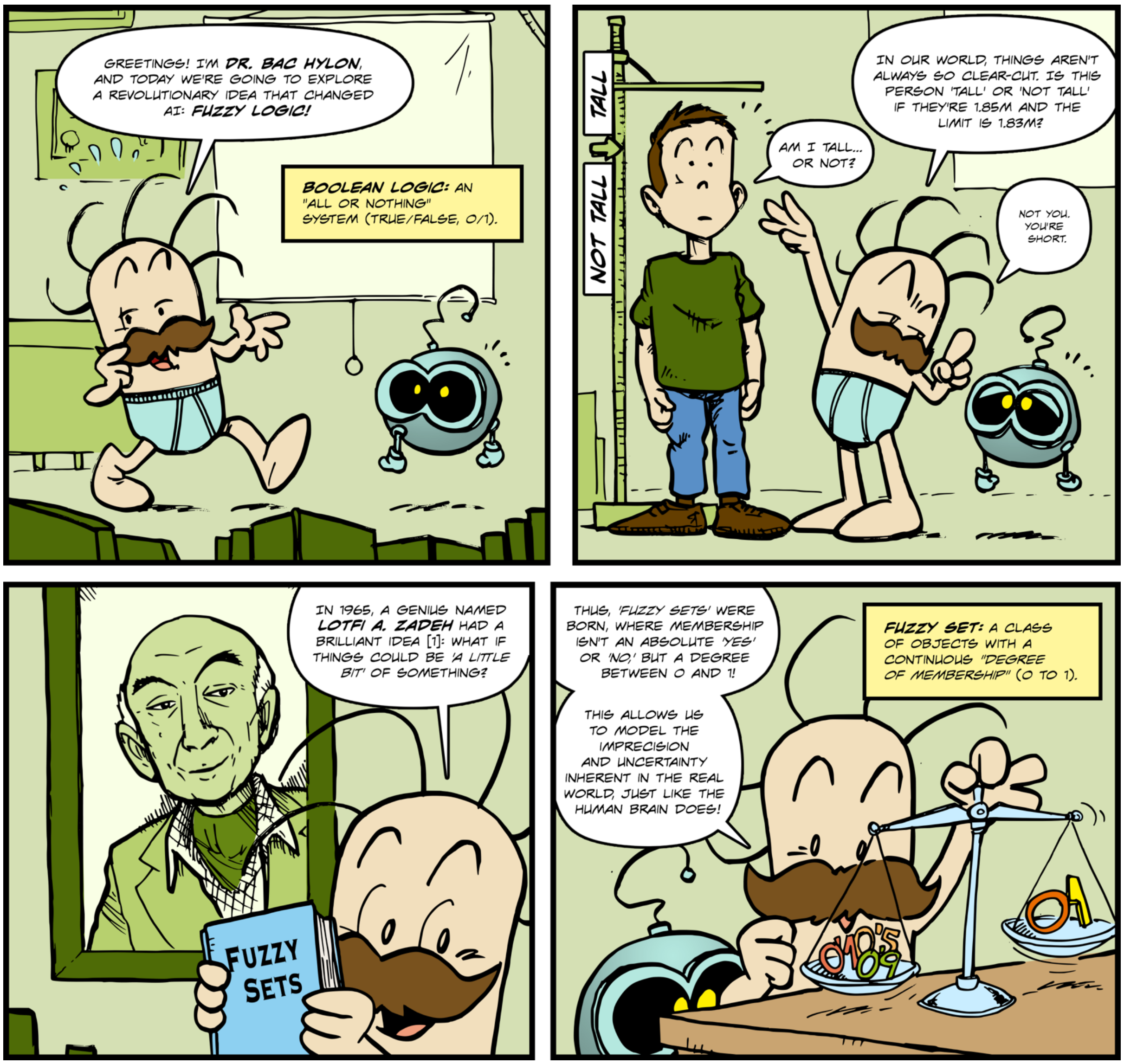
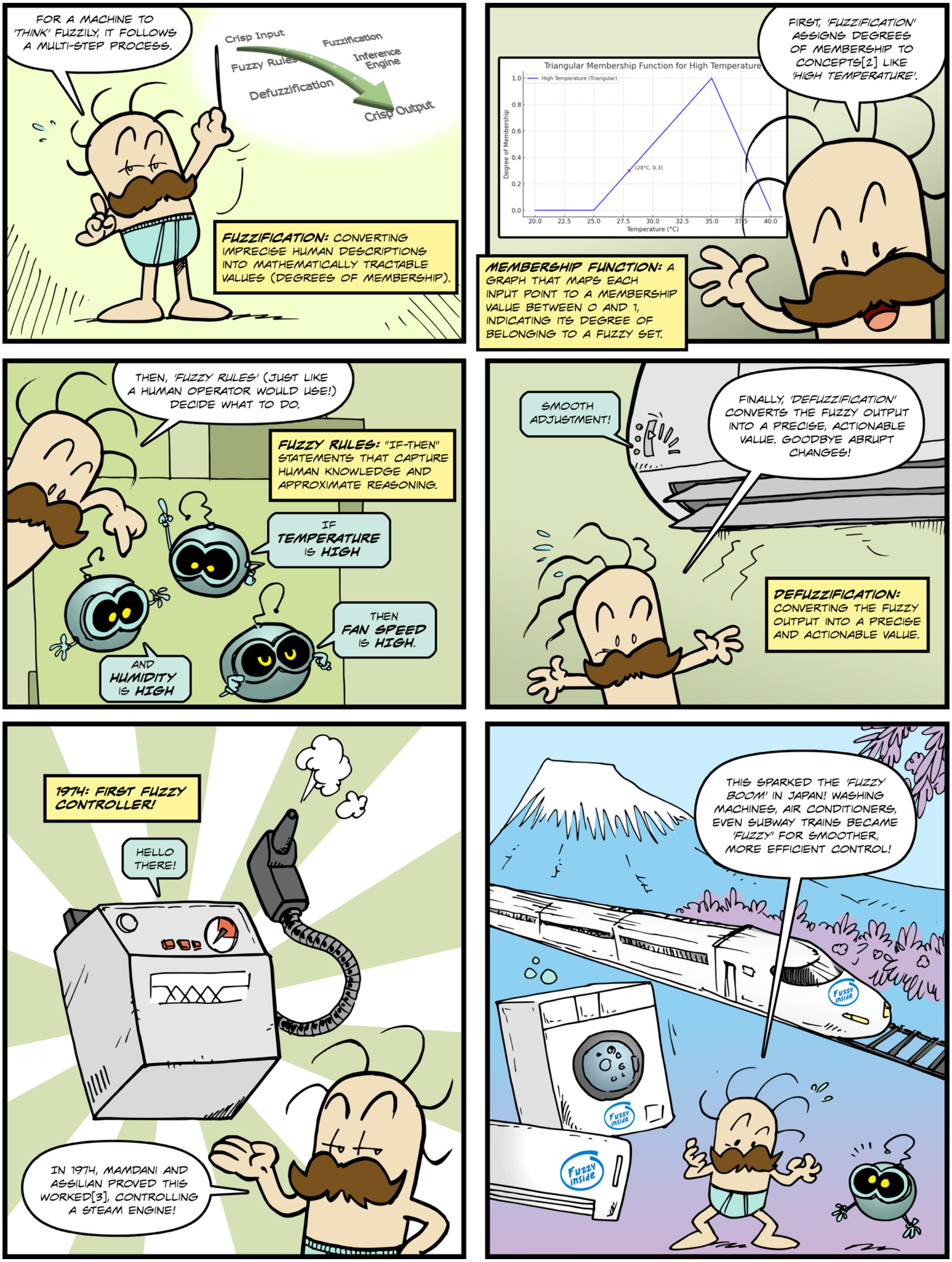
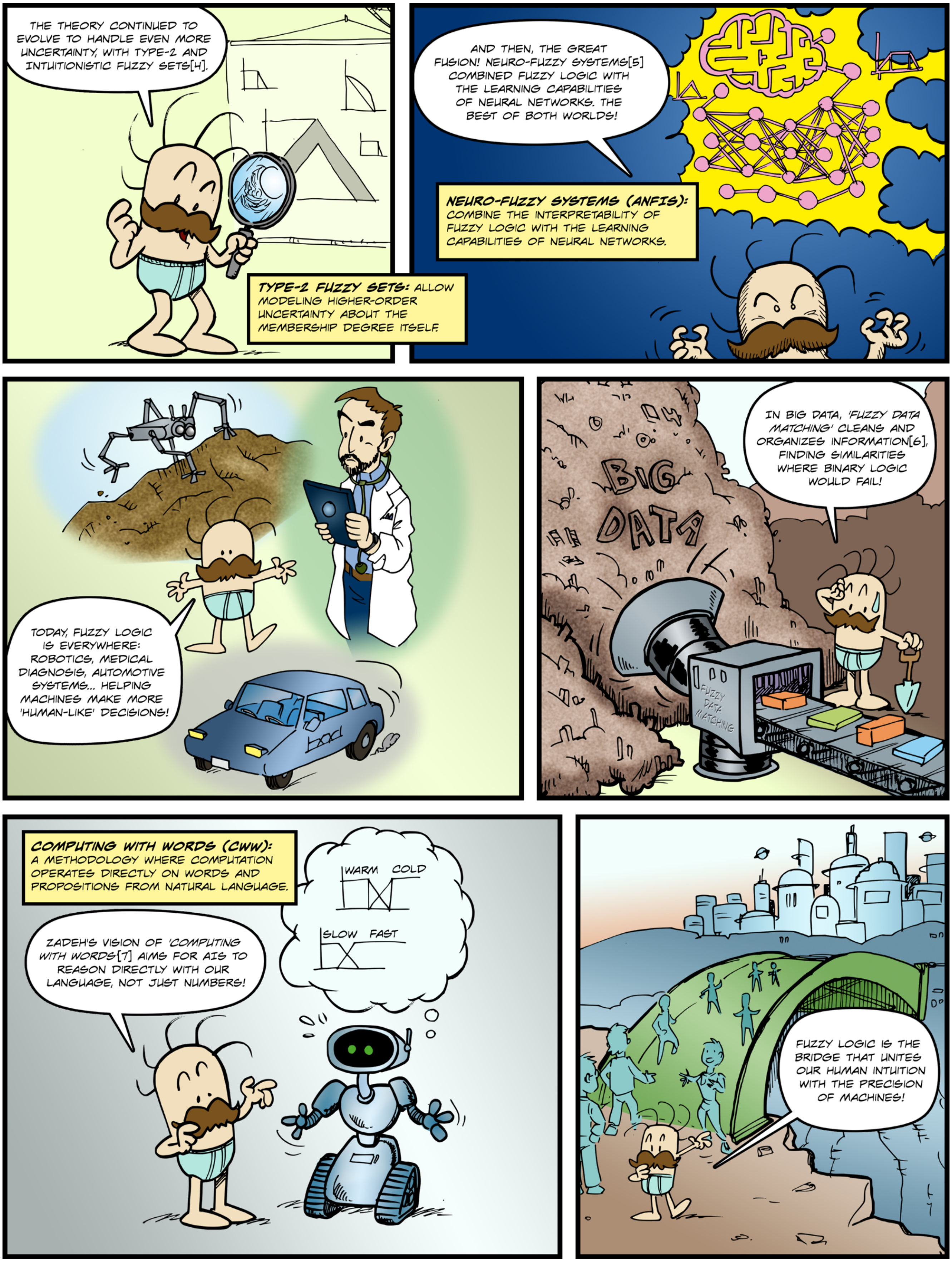
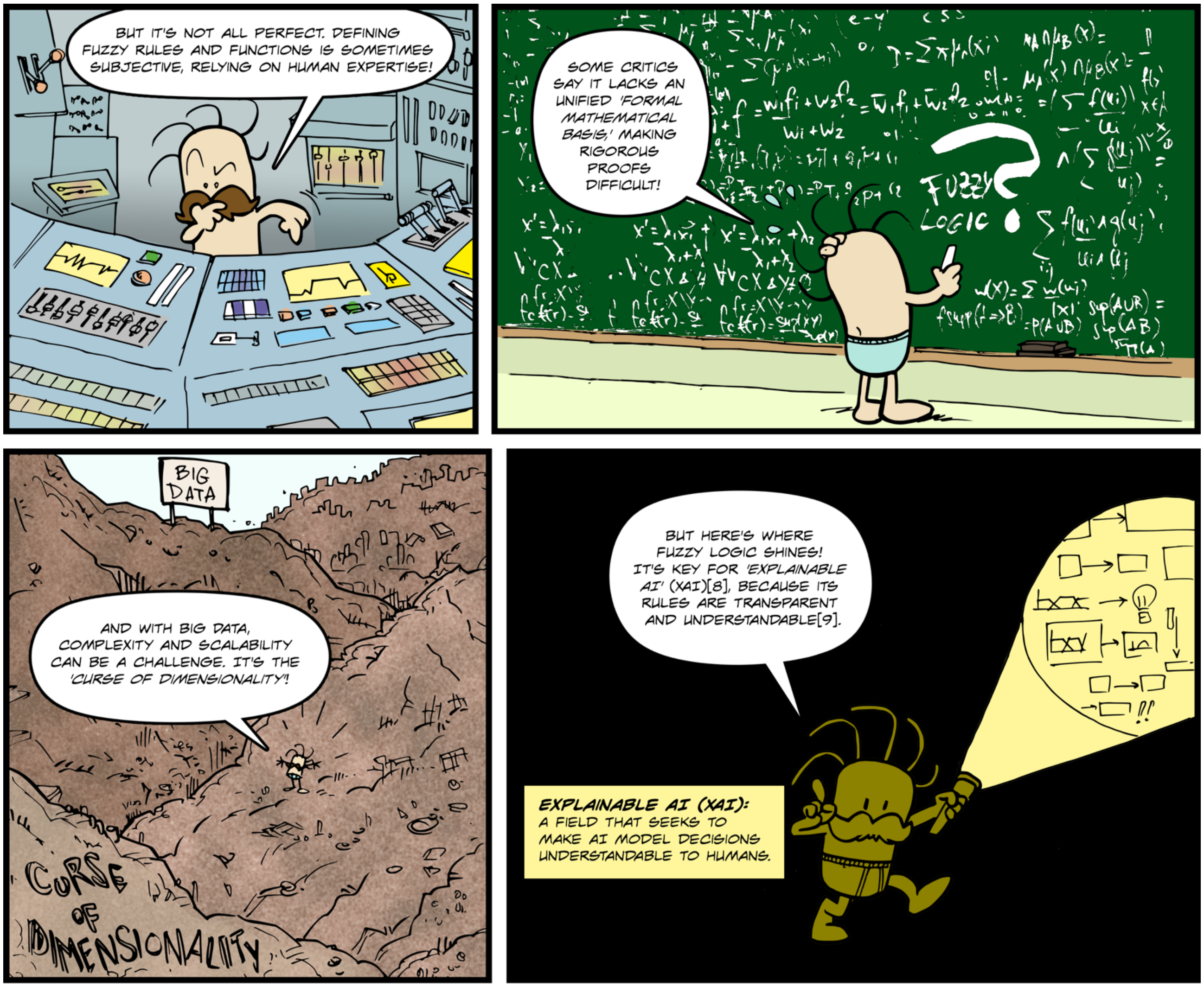
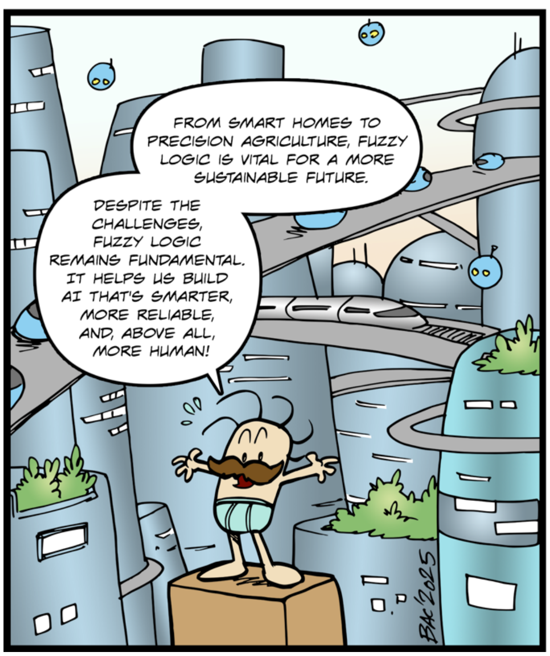



FOR A MACHINE TO 'THINK' FUZZILY, IT FOLLOWS A MULTI-STEP PROCESS.
Crisp Input
Fuzzification
Fuzzy Rules
Inference Engine
Defuzzification
Crisp Output
FUZZIFICATION: CONVERTING IMPRECISE HUMAN DESCRIPTIONS INTO MATHEMATICALLY TRACTABLE VALUES (DEGREES OF MEMBERSHIP).
FIRST, 'FUZZIFICATION' ASSIGNS DEGREES OF MEMBERSHIP TO CONCEPTS[2] LIKE 'HIGH TEMPERATURE'.
Triangular Membership Function for High Temperature
High Temperature (Triangular)
(28°C, 0.3)
Degree of Membership
Temperature (°C)
MEMBERSHIP FUNCTION: A GRAPH THAT MAPS EACH INPUT POINT TO A MEMBERSHIP VALUE BETWEEN 0 AND 1, INDICATING ITS DEGREE OF BELONGING TO A FUZZY SET.
THEN, 'FUZZY RULES' (JUST LIKE A HUMAN OPERATOR WOULD USE!) DECIDE WHAT TO DO.
FUZZY RULES: "IF-THEN" STATEMENTS THAT CAPTURE HUMAN KNOWLEDGE AND APPROXIMATE REASONING.
IF TEMPERATURE IS HIGH
AND HUMIDITY IS HIGH
THEN FAN SPEED IS HIGH.
SMOOTH ADJUSTMENT!
FINALLY, 'DEFUZZIFICATION' CONVERTS THE FUZZY OUTPUT INTO A PRECISE, ACTIONABLE VALUE. GOODBYE ABRUPT CHANGES!
DEFUZZIFICATION: CONVERTING THE FUZZY OUTPUT INTO A PRECISE AND ACTIONABLE VALUE.
1974: FIRST FUZZY CONTROLLER!
HELLO THERE!
IN 1974, MAMDANI AND ASSILIAN PROVED THIS WORKED[3], CONTROLLING A STEAM ENGINE!
THIS SPARKED THE 'FUZZY BOOM' IN JAPAN! WASHING MACHINES, AIR CONDITIONERS, EVEN SUBWAY TRAINS BECAME 'FUZZY' FOR SMOOTHER, MORE EFFICIENT CONTROL!
Fuzzy inside
Fuzzy inside
Fuzzy inside


THE THEORY CONTINUED TO EVOLVE TO HANDLE EVEN MORE UNCERTAINTY, WITH TYPE-2 AND INTUITIONISTIC FUZZY SETS[4].
TYPE-2 FUZZY SETS: ALLOW MODELING HIGHER-ORDER UNCERTAINTY ABOUT THE MEMBERSHIP DEGREE ITSELF.
AND THEN, THE GREAT FUSION! NEURO-FUZZY SYSTEMS[5] COMBINED FUZZY LOGIC WITH THE LEARNING CAPABILITIES OF NEURAL NETWORKS. THE BEST OF BOTH WORLDS!
NEURO-FUZZY SYSTEMS (ANFIS): COMBINE THE INTERPRETABILITY OF FUZZY LOGIC WITH THE LEARNING CAPABILITIES OF NEURAL NETWORKS.
TODAY, FUZZY LOGIC IS EVERYWHERE: ROBOTICS, MEDICAL DIAGNOSIS, AUTOMOTIVE SYSTEMS... HELPING MACHINES MAKE MORE 'HUMAN-LIKE' DECISIONS!
BIG DATA
IN BIG DATA, 'FUZZY DATA MATCHING' CLEANS AND ORGANIZES INFORMATION[6], FINDING SIMILARITIES WHERE BINARY LOGIC WOULD FAIL!
FUZZY DATA MATCHING
COMPUTING WITH WORDS (CWW): A METHODOLOGY WHERE COMPUTATION OPERATES DIRECTLY ON WORDS AND PROPOSITIONS FROM NATURAL LANGUAGE.
WARM COLD
SLOW FAST
ZADEH'S VISION OF 'COMPUTING WITH WORDS[7] AIMS FOR AIS TO REASON DIRECTLY WITH OUR LANGUAGE, NOT JUST NUMBERS!
FUZZY LOGIC IS THE BRIDGE THAT UNITES OUR HUMAN INTUITION WITH THE PRECISION OF MACHINES!





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References

[1] Zadeh, L. A. (1965). Fuzzy sets. *Information and control*, *8*(3), 338-353.

[2] Zadeh, L. A. (1975). The concept of a linguistic variable and its application to approximate reasoning—I. *Information sciences*, *8*(3), 199-249.

[3] Mamdani, E. H. (1974). Application of fuzzy algorithms for control of simple dynamic plant. In *Proceedings of the institution of electrical engineers* (Vol. 121, No. 12, pp. 1585-1588). IEE.

[4] Atanassov, K.T. (1986). Intuitionistic fuzzy sets. Fuzzy Sets Syst. 20, 1 (August, 1986), 87–96.

[5] Jang, J. S. (1993). ANFIS: adaptive-network-based fuzzy inference system. *IEEE transactions on systems, man, and cybernetics*, *23*(3), 665-685.

[6] Subramanian, B. , Mishra, A. , Bharathi V, R. , Mandala, G. , Kathamuthu, N. D. and Srithar, S. (2025). Big data and fuzzy logic for demand forecasting in supply chain management: A data-driven approach. *Journal of Fuzzy Extension and Applications*, *6*(2), 260-283.

[7] Zadeh, L. A. (1996). Fuzzy logic= computing with words. *IEEE transactions on fuzzy systems*, *4*(2), 103-111.

[8] Fernandez, A., Herrera, F., Cordon, O., del Jesus, M. J., & Marcelloni, F. (2019). Evolutionary fuzzy systems for explainable artificial intelligence: Why, when, what for, and where to?. *IEEE Computational intelligence magazine*, *14*(1), 69-81.

[9] Delgado, M., Marín, N., Sánchez, D., & Vila, M. A. (2003). Fuzzy association rules: general model and applications. *IEEE transactions on Fuzzy Systems*, *11*(2), 214-225.

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